How Many Multiplicative Magic Squares are There?

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It takes any mathematician worth his or her salt only a few moments (and any reasonably bright child even less!) to to realize that

$$\begin{pmatrix} 13 & 9 & 2 & 6 \\ 7 & 4 & 11 & 8 \\ 0 & 3 & 12 & 15 \\ 10 & 14 & 5 & 1 \end{pmatrix}$$

is an additive magic square. But how about

$$\begin{pmatrix} 90 & 10 & 3 & 27 \\ 135 & 9 & 30 & 2 \\ 1 & 15 & 18 & 270 \\ 6 & 54 & 45 & 5 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 486 & 6 & 9 & 729 \\ 2187 & 81 & 54 & 2 \\ 1 & 27 & 162 & 4374 \\ 18 & 1458 & 243 & 3 \end{pmatrix}?$$

These are both multiplicative magic squares, $n \times n$ matrices of integers such that the product along each row, each column and front and back diagonal is the same. This is, of course, simply the definition of an additive magic square with *sum* having been replaced by *product*. It is well known how many different additive magic squares there are, at least for $n \leq 5$. In this note we use those results to compute the number of multiplicative magic squares for $n \leq 5$.

As the square of all 1's is trivially both a multiplicative magic square (henceforth referred to as a MMS) and an additive magic square (an AMS), one commonly requires that the square be **normal**. In an additive square this means that the entries are the numbers $0, 1, ..., n^2 - 1$. (Actually, it is customary to let the entries run from 1 to n^2, but reducing each entry by 1 will make this note clearer.) Further, squares that are simply rotations or reflections of one another are treated as the same square. With these restrictions, the number of AMS's of various sizes is given by

size	number of AMS's
1×1	1
2×2	0
3×3	1
4×4	880
5×5	275,305,224
6×6	unknown

Number of Additive Magic Squares

With our restrictions, (0) is the only 1×1 magic square and the reader can easily show that there are no 2×2 AMS's. The unique 3×3 additive magic square, again with our restrictions, is

$$\begin{pmatrix} 7 & 2 & 3 \\ 0 & 4 & 8 \\ 5 & 6 & 1 \end{pmatrix}.$$

It is referred to as the *Lo Shu* and is often credited with mystical connections. The set of fundamental 4×4 AMS's was first published in 1693 by Frénicle de Bessey [1] and has been carefully studied by H. E. Dudeney and Dame Kathleen Ollerenshaw. [4], among others. Finally, in 1973 Richard Schroeppel computed the number of 5×5 AMS's [2].

Before beginning our count of the number of MMS's, we must first determine what normal means for a MMS. The normality condition for an AMS implies that the entries are distinct positive integers that are consecutive and as small as possible. Since the entries of a MMS are all divisors of some integer, normality should imply that the entries are all the distinct divisors of some integer P. Further, if we were to allow

$$\begin{pmatrix} 2^7 & 2^2 & 2^3 \\ 2^0 & 2^4 & 2^8 \\ 2^5 & 2^6 & 2^1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3^7 & 3^2 & 3^3 \\ 3^0 & 3^4 & 3^8 \\ 3^5 & 3^6 & 3^1 \end{pmatrix}$$

to count as two MMS's then once we have one, we would have infinitely many. So we will also require that the largest entry P of the square to be as small as possible, in a reasonable sort of way. For example, with $p < q < r$ prime numbers, the three squares

$$\begin{pmatrix} pq^2r & pr & q & q^3 \\ q^3r & q^2 & pqr & p \\ 1 & qr & pq^2 & pq^3r \\ pq & pq^3 & q^2r & r \end{pmatrix}, \begin{pmatrix} pq^2r & pr & q & q^3 \\ pq^3 & q^2 & pqr & r \\ 1 & pq & q^2r & pq^3r \\ qr & q^3r & pq^2 & p \end{pmatrix} \text{ and}$$

$$\begin{pmatrix} pq^5 & pq & q^2 & q^6 \\ q^7 & q^4 & pq^3 & p \\ 1 & q^3 & pq^4 & pq^7 \\ pq^2 & pq^6 & q^5 & q \end{pmatrix}$$

are counted as distinct MMS's, even though the second square is created from the first by swapping the roles of p and r, and the third from the first by replacing q^ir^j by q^{2i+j}. (It was the first and last of these that were specialized to give the examples in the introduction.)

Thus, we define a normal MMS to be a MMS whose entries are the set of distinct divisors of some integer P, with P having distinct prime divisors of P that are as small as possible. With this taken care of, we may state our main result.

Theorem: The number of normal multiplicative magic squares of size $n \times n$ is 1, 0, 2, 5264 and $276,942,724$, for $n = 1, 2, 3, 4$ and 5, respectively.

The keys to our work are two-fold. First, there clearly is a strong connection between the set of AMS's and the set of MMS's. If $M = (m_{ij})$ is a MMS, and N is the square whose entries are $n_{ij} = ord_p(m_{ij})$, where $ord_p(m)$ is the power of p appearing in the prime factorization of m, then N is an AMS. We will write this as $N = ord_p(M)$. (Notice that M being normal does not imply that N is.) Similarly, for an AMS $N = (n_{ij})$, the square M whose entries are $m_{ij} = p^{n_{ij}}$ is a MMS. We will write $M = exp_p(N)$. (In this case normality is preserved.)

Second, for a fixed MMS there are stringent bounds on the prime factorizations of its largest entry entry P. The reader will easily prove the following result.

Lemma: Suppose $P = \prod_i {p_i}^{e_i}$ is the prime factorization of the the largest entry in an $n \times n$ MMS. Then $\prod_i (e_i + 1) = n^2$.

Now we may consider the various sizes.

1×1 Multiplicative Magic Squares

Clearly from our normality restrictions, $(\,1\,)$ is the only 1×1 MMS.

2×2 Multiplicative Magic Squares

When $n = 2$ the requirement on $P = \prod p_i^{e_i}$ is $\prod (e_i + 1) = 4$. So $P = pq$ or $P = p^3$. (Normality requires that $p = 2$ and $q = 3$, but using p and q will allow to more easily concentrate on the important items, the exponents.)

If M is a normal MMS with either $P = pq$ or $P = p^2$, and we let $N = ord_p(M)$, then N is an 2×2 AMS whose entries are not all equal. However there are no such squares. Hence, there are no 2×2 MMS's.

3×3 Multiplicative Magic Squares

We must have $\prod (e_i + 1) = 9$, which has only two solutions. First, if $e_1 = 8$, since there is one normal AMS, there will be one normal MMS with largest entry p^8. It is $exp_p(L)$, where L is the Lo Shu square.

Next, suppose M is a MMS with $e_1 = e_2 = 2$. Then the row product of M is $p^3 q^3$ and each entry of M has the form $p^i q^j$ for $0 \le i, j \le 2$. Here we let N be the square formed from M by replacing $p^i q^j$ by $3i + j$. Since the rows, columns and diagonals of M contain 3 copies of each of p and q, the row, column and diagonal sums of N must be $3 \times 3 + 3 = 12$. Hence N is an AMS. It is similarly easy to check that N is normal. Thus $N = L$ (up to rotation and reflection).

Conversely, writing entries of the Lo Shu in base 3 gives

$$\begin{pmatrix} 21 & 02 & 10 \\ 00 & 11 & 22 \\ 12 & 20 & 01 \end{pmatrix}.$$

This provides the MMS

$$\begin{pmatrix} p^2q^1 & p^0q^2 & p^1q^0 \\ p^0q^0 & p^1q^1 & p^2q^2 \\ p^1q^2 & p^2q^0 & p^0q^1 \end{pmatrix} = \begin{pmatrix} p^2q^1 & q^2 & p \\ 1 & pq & p^2q^2 \\ pq^2 & p^2 & q \end{pmatrix}.$$

The key here is that the **coefficient squares**,

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix},$$

are both additive magic squares that are composed of three copies each of 0, 1 and 2.

Therefore, up to rotation and reflection there is exactly one 3×3 MMS of form p^8 and and exactly one of the form p^2q^2. So there are two 3×3 normal MMS's.

4×4 Multiplicative Magic Squares

The equation $\prod(e_i + 1) = 16$ has eight different solutions, giving eight possible types of 4×4 MMS's. We list these types, their solutions, as well as the largest entry in such a square and the row product of that square.

type	solution	largest entry	row product
(1)	$e_1 = 15$	$P = p^{15}$	p^{30}
(2)	$e_1 = 1, e_2 = 7$	$P = pq^7$	p^2q^{14}
(3)	$e_1 = 7, e_2 = 1$	$P = p^7q$	$p^{14}q^2$
(4)	$e_1 = 3, e_2 = 3$	$P = p^3q^3$	p^6q^6
(5)	$e_1 = 1, e_2 = 1, e_3 = 3$	$P = pqr^3$	$p^2q^2r^6$
(6)	$e_1 = 1, e_2 = 3, e_3 = 1$	$P = pq^3r$	$p^2q^6r^2$
(7)	$e_1 = 3, e_2 = 1, e_3 = 1$	$P = p^3qr$	$p^6q^2r^2$
(8)	$e_1 = 1, e_2 = 1, e_3 = 1, e_4 = 1$	$P = pqrs$	$p^2q^2r^2s^2$

Here $p < q < r < s$ are primes (and normality implies they are 2, 3, 5 and 7, resp.).

We first show that to each MMS (assuming any exist) there is a naturally associated AMS. Then we see to what degree this mapping is bijective.

Proposition: For each 4×4 normal MMS there is a corresponding normal AMS.

Proof: Consider the following maps from the set of 4×4 normal MMS's to the set of 4×4 matrices, one map for each type. The maps are entry-to-entry.

(1) $M \to N : m = p^i \mapsto n = i$
(2) $M \to N : m = p^iq^j \mapsto n = 8i + j$.
(3) $M \to N : m = p^iq^j \mapsto n = 2i + j$.
(4) $M \to N : m = p^iq^j \mapsto n = 4i + j$.
(5) $M \to N : m = p^iq^jr^k \mapsto n = 8i + 4j + k$.
(6) $M \to N : m = p^iq^jr^k \mapsto n = 8i + 2j + k$.
(7) $M \to N : m = p^iq^jr^k \mapsto n = 4i + 2j + k$.
(8) $M \to N : m = p^iq^jr^ks^l \mapsto n = 8i + 4j + 2k + l$.

It is easy to see that these are each maps into the set of 4×4 normal AMS's, and that each map is injective. $\square$

For example, $\begin{pmatrix} 1 & rs & pq & pqrs \\ pqrs & pqs & r & p \\ pqr & q & prs & s \\ ps & pr & qs & qr \end{pmatrix}$ and $\begin{pmatrix} 1 & q^3 & p^3 & p^3q^3 \\ pq^3 & p^3q & q^2 & p^2 \\ p^3q^2 & p^2 & p^2q^3 & q \\ p^2q & p^2q^2 & pq & pq^2 \end{pmatrix}$, of types (8) and (4), respectively, are both mapped to $\begin{pmatrix} 0 & 3 & 12 & 15 \\ 7 & 13 & 2 & 8 \\ 14 & 4 & 11 & 1 \\ 9 & 10 & 5 & 6 \end{pmatrix}$. On the other hand, $\begin{pmatrix} 0 & 13 & 2 & 15 \\ 14 & 10 & 5 & 1 \\ 9 & 3 & 12 & 6 \\ 7 & 4 & 11 & 8 \end{pmatrix}$ is an AMS that does not come from any MMS.

(It turns out that the AMS's with diagonals composed of $\{0, 8, 12, 10\}$ and $\{3, 5, 7, 15\}$ are exactly those squares that are not the image of any MMS. In [3] it was suggested that the map we have listed in the proof as (8) was a bijection. Clearly this is not the case.)

To see which AMS's are the image of MMS's, i.e., which MMS's actually exist, first notice that the maps in the proof of the Proposition give eight ways to enumerate the numbers from 0 to 15:

(1) i, for $0 \le i < 16$
(2) $8i + j$, for $0 \le i < 2$ and $0 \le j < 8$
(3) $2i + j$, for $0 \le i < 8$ and $0 \le j < 2$
(4) $4i + j$, for $0 \le i, j < 4$
(5) $8i + 4j + k$, for $0 \le i, j < 2$ and $0 \le k < 4$
(6) $8i + 2j + k$, for $0 \le i, k < 2$ and $0 \le j < 4$
(7) $4i + 2j + k$, for $0 \le j, k < 2$ and $0 \le i \le 4$
(8) $8i + 4j + 2k + l$, for $0 \le i, j, k, l < 2$

An AMS may be broken into coefficient squares based on any of these enumerations. For example, using method (5), we have

$$\begin{pmatrix} 15 & 13 & 0 & 2 \\ 3 & 1 & 14 & 12 \\ 8 & 10 & 5 & 7 \\ 4 & 6 & 11 & 9 \end{pmatrix} \mapsto \begin{pmatrix} 113 & 111 & 000 & 002 \\ 003 & 001 & 112 & 110 \\ 100 & 102 & 011 & 013 \\ 010 & 012 & 103 & 101 \end{pmatrix}$$

A quick examination will conclude that the three corresponding coefficient squares are each a (non-normal) additive magic square, and so this AMS leads to the MMS

$\begin{pmatrix} pqr^3 & pqr & 1 & r^2 \\ r^3 & r & pqr^2 & pq \\ p & pr^2 & qr & qr^3 \\ q & qr^2 & pr^3 & pr \end{pmatrix}$. However, using enumeration (3) gives

$$\begin{pmatrix} 15 & 13 & 0 & 2 \\ 3 & 1 & 14 & 12 \\ 8 & 10 & 5 & 7 \\ 4 & 6 & 11 & 9 \end{pmatrix} \mapsto \begin{pmatrix} 71 & 61 & 00 & 10 \\ 11 & 01 & 70 & 60 \\ 40 & 50 & 21 & 31 \\ 20 & 30 & 51 & 41 \end{pmatrix}$$

which fails to have additively magic coefficient squares.

So we say that an AMS is a specific type if, when using the enumeration of that type, its coefficient squares are each additively magic. Thus the AMS in the previous paragraph is type (5) but not type (3).

It is easy to see the following.

Lemma: Of the normal 4×4 AMS's:

(1) All are type (1).
(2) If a square is type (8), then it is also type (1), (2), ..., and (7).
(3) Type (5) implies types (2) and (4).
(4) Type (6) implies types (2) and (3).
(5) Type (5) implies types (2) and (4).

A simple computer program and a list of of the 880 normal magic squares, obtainable from various web sites, quickly give the following numbers.

Proposition: Of the 880 4×4 normal magic squares:

(1) 880 matrices are type (1).
(2) 712 matrices are type (2).
(3) 712 matrices are type (3).
(4) 656 matrices are type (4).
(5) 592 matrices are type (5).
(6) 592 matrices are type (6).
(7) 592 matrices are type (7).
(8) 528 matrices are type (8).

Corollary: There are 5264 4×4 normal MMS's.

5×5 **Multiplicative Magic Squares**

Since 5 is a prime number, this case is very similar to the 3×3 case. Here $\prod(e_i + 1) = 25$, so we have $e_1 = 24$ or $e_1 = e_2 = 4$. Thus to each 5×5 normal MMS M there is a naturally corresponding 5×5 normal AMS N, via the mapping

$$M \to N : m = p^i \mapsto n = i \qquad \text{if } e_1 = 24\text{, i.e., } 0 \le i \le 24$$
$$M \to N : m = p^i q^i \mapsto n = 5i + j \qquad \text{if } e_1 = e_2 = 4\text{, i.e., } 0 \le i, j \le 4.$$

Conversely, each normal AMS's gives a MMS by $N \to M : n \mapsto m = p^n$. Further, a simple, if lengthy, computation shows that of the normal 5×5 AMS's, exactly $1,637,500$ lead to MMS's using the mapping $N \to M : n = 5i + j \mapsto m = p^i q^j$, that is, there are this many MMS's with coefficient squares that are magic using the enumeration $5i + j$.

For example,

$$\begin{pmatrix} 0 & 19 & 17 & 23 & 1 \\ 14 & 7 & 6 & 15 & 18 \\ 20 & 5 & 24 & 2 & 9 \\ 22 & 16 & 3 & 8 & 11 \\ 4 & 13 & 10 & 12 & 21 \end{pmatrix} = \begin{pmatrix} 00 & 34 & 32 & 43 & 01 \\ 24 & 02 & 01 & 30 & 13 \\ 40 & 10 & 44 & 02 & 14 \\ 42 & 31 & 03 & 13 & 21 \\ 04 & 23 & 20 & 22 & 41 \end{pmatrix}_{\text{base } 5}$$

provides two different MMS's, one by direct the exponentiation of its coefficients, and one by using the coefficient square entries as exponents. However,

$$\begin{pmatrix} 0 & 18 & 22 & 19 & 1 \\ 23 & 7 & 2 & 17 & 11 \\ 13 & 5 & 24 & 6 & 12 \\ 20 & 14 & 3 & 8 & 15 \\ 4 & 16 & 9 & 10 & 21 \end{pmatrix} = \begin{pmatrix} 00 & 33 & 42 & 34 & 01 \\ 43 & 12 & 02 & 32 & 21 \\ 23 & 10 & 44 & 11 & 22 \\ 40 & 24 & 03 & 13 & 30 \\ 04 & 31 & 14 & 20 & 41 \end{pmatrix}_{\text{base } 5}$$

leads to only one, since the third and and fifth columns of its coefficient matrices do not sum to 10.

We have proven

Proposition: There are $276,942,724$ 5×5 normal MMS's.

References

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