Readings from the Invention of the Calculus Integral Program Reading

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1 Archimedes: Quadrature of the Parabola

Archimedes to Dositheus greeting.[1]

When I heard that Conon, who was my friend in his lifetime, was dead, but that you were acquainted with Conon and withal versed in geometry, while I grieved for the loss not only of a friend but of an admirable mathematician, I set myself the task of communicating to you, as I had intended to send to Conon, a certain geometrical theorem which had not been investigated before but has now been investigated by me, and which I first discovered by means of mechanics and then exhibited by means of geometry. Now some of the earlier geometers tried to prove it possible to find a rectilineal area equal to a given circle and a given segment of a circle; and after that they endeavoured to square the area bounded by the section of the whole cone and a straight line, assuming lemmas not easily conceded, so that it was recognised by most people that the problem was not solved. But I am not aware that any one of my predecessors has attempted to square the segment bounded by a straight line and a section of a right-angled cone [a parabola], of which problem I have now discovered the solution. For it is here shown that every segment bounded by a straight line and a section of a right-angled cone [a parabola] is four-thirds of the triangle which has the same base and equal height with the segment, and for the demonstration of this property the following lemma is assumed: that the excess by which the greater of (two) unequal areas exceeds the less can, by being added to itself, be made to exceed any given finite area. The earlier geometers have also used this lemma; for it is by the use of this same lemma that they have shown that circles are to one another in the duplicate ratio of their diameters, and that spheres are to one another in the triplicate ratio of their diameters, and further that every pyramid is one third part of the prism which has the same base with the pyramid and equal height; also, that every cone is one third part of the cylinder having the same base as the cone and equal height they proved by assuming a certain lemma similar to that aforesaid. And, in the result, each of the aforesaid theorems has been accepted no less than those proved without the lemma. As therefore my work now published has satisfied the same test as the propositions referred to, I have written out the proof and send it to you, first as investigated by means of mechanics, and afterwards too as demonstrated by geometry. Prefixed are, also, the elementary propositions in conics which are of service in the proof. Farewell.

... Proposition 20.

If \( Qq \) be the base, and \( P \) the vertex, of a parabolic segment, then the triangle \( PQq \) is greater than half the segment \( PQq \).

For the chord \( Qq \) is parallel to the tangent at \( P \), and the triangle

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PQq is half the parallelogram formed by Qq, the tangent at P, and the diameters through Q, q.

Therefore the triangle PQq is greater than half the segment.

Cor. It follows that it is possible to inscribe in the segment a polygon such that the segments left over are together less than any assigned area.

... 

Proposition 23.

Given a series of areas A, B, C, D, ..., Z, of which A is the greatest, and each is equal to four times the next in order, then

\[ A + B + C + ... + Z + \frac{1}{3}Z = \frac{4}{3}A. \]

Take areas b, c, d, ... such that \( b = \frac{1}{3}B, c = \frac{1}{3}C, d = \frac{1}{3}D, \) and so on. Then, since \( b = \frac{1}{3}B \) and \( B = \frac{1}{4}A, A + b = \frac{1}{3}A. \) Similarly \( C + c = \frac{1}{3}B. \) Therefore

\[ B + C + D + ... + Z + b + c + d + ... + z = \frac{1}{3}(A + B + C + ... + Y). \]

But

\[ b + c + d + ... + y = \frac{1}{3}(B + C + D + ... + Y). \]

Therefore, by subtraction,

\[ B + C + D + ... + Z + z = \frac{1}{3}A \]

or

\[ A + B + C + D + ... + Z + \frac{1}{3}Z = \frac{4}{3}A. \]

Proposition 24.

Every segment bounded by a parabola and a chord Qq is equal to four-thirds of the triangle which has the same base as the segment and equal height.

Suppose \( K = \frac{4}{3}\Delta PQq, \) where P is the vertex of the segment; and we have then to prove that the area of the segment is equal to \( K. \)

\[ 2 \text{ Heath: The algebraical equivalent of this result is of course } 1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + ... + \left(\frac{1}{4}\right)^{n-1} = \frac{4}{3} - \frac{1}{3}\left(\frac{1}{4}\right)^n = \frac{1 - (\frac{1}{4})^n}{1 - \frac{1}{4}}. \]
For, if the segment be not equal to $K$, it must either be greater or less.

I. Suppose the area of the segment greater than $K$.

If then we inscribe in the segments cut off by $PQ$, $Pq$ triangles which have the same base and equal height, i.e. triangles with the same vertices $R$, $r$ as those of the segments, and if in the remaining segments we inscribe triangles in the same manner, and so on, we shall finally have segments remaining whose sum is less than the area by which the segment $PQq$ exceeds $K$.

Therefore the polygon so formed must be greater than the area $K$; which is impossible, since [Prop. 23]

$$A + B + C + ... + Z < \frac{4}{3}A,$$

where $A = \Delta PQq$.

Thus the area of the segment cannot be greater than $K$.

II. Suppose, if possible, that the area of the segment is less than $K$.

If then $\Delta PQq = A$, $B = \frac{1}{4}A$, $C = \frac{1}{4}B$, and so on, until we arrive at an area $X$ such that $X$ is less than the difference between $K$ and the segment, we have

$$A + B + C + ... + X + \frac{1}{3}X = \frac{4}{3}A = K. \text{ [Prop. 23]}$$

Now, since $K$ exceeds $A + B + C + \ldots + X$ by an area less than $X$, and the area of the segment by an area greater than $X$, it follows that

$$A + B + C + ... + Z > (\text{the segment});$$

which is impossible, by Prop. 22 above. Hence the segment is not less than $K$. Thus, since the segment is neither greater nor less than $K$,

$$(\text{area of segment } PQq) = K = \frac{4}{3}\Delta PQq.$$
2 Tangents: How to find tangent lines?

**Euclid:** Definition III.2. A straight line is said to touch a circle which, meeting the circle and being produced, does not cut the circle.

Proposition III.16. The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle.

**Apollonius:** Proposition I.33: If on a parabola some point is taken, and from it an ordinate is dropped to the diameter, and, to the straight line cut off by it on the diameter from the vertex, a straight line in the same straight line from its extremity is made equal, then the straight line joined from the point thus resulting to the point taken will touch the section.

Proposition I.34: If on an hyperbola or ellipse or circumference of a circle some point is taken, and if from it a straight line is dropped ordinatewise to the diameter, and if the straight lines which the ordinatewise line cuts off from the ends of the figures’ tranverse side have to each other a ratio which other segments of the transverse side have to each other, so that the segments from the vertex are corresponding, then the straight line joining the point taken on the transverse side and that taken on the section will touch the section.

2.1 Pierre de Fermat: *On Maxima and Minima*

(1) On a Method for the Evaluation of Maxima and Minima

The whole theory of evaluation of maxima and minima presupposes two unknown quantities and the following rule:

Let $a$ be any unknown of the problem (which is in one, two, or three dimensions, depending on the formulation of the problem). Let us indicate the maximum or minimum by $a$ in terms which could be of any degree. We shall now replace the original unknown $a$ by $a + e$ and we shall express thus the maximum or minimum quantity in terms of $a$ and $e$ involving any degree. We shall adequate, to use Diophantus’ term, the two expressions of the maximum or minimum quantity and we shall take out their common terms. Now it will turn out that both sides contain terms in $e$ or its powers. We shall divide all terms by $e$, or by a higher power of $e$, so that $e$ will be completely removed from at least one of the terms.

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5 Pierre de Fermat (1601–1665), Struik: “This paper was sent by Fermat to Father Marin Merenne, who forwarded it to Descartes. Descartes received it January 1638. It became the subject of a polemic discussion between him and Fermat.” Taken from *A Source Book in Mathematics, 1200-1800*, edited by Dirk Jan Struik, Harvard University Press, Cambridge, 1969, pages 222–224.
We suppress then all the terms in which \( e \) or one of its powers will still appear, and we shall equate the others; or, if one of the expressions vanishes, we shall equate, which is the same thing, the positive and negative terms. The solution of this last equation will yield the value of \( a \) which will lead to the maximum or minimum, by using again the original expression.\(^6\)

Here is an example:

To divide the segment \( AC \) at \( E \) so that \( AE \times EC \) may be maximum.

We write \( AC = b \); let \( a \) be one of the segments, so that the other will be \( b - a \), and the product, the maximum of which is to be found, will be \( ba - a^2 \). Let now \( a + ea \) be the first segment of \( b \); the second will be \( b - a - e \), and the product of the segments, \( ba - a^2 + be - 2ae - e^2 \); this must be equated with the preceding: \( ba - a^2 \). Suppressing common terms: \( be \sim 2ae + e \). Suppressing \( e \): \( b = 2a \). To solve the problem we must consequently take the half of \( b \).

We can hardly expect a more general method.

ON THE TANGENTS OF CURVES

We use the preceding method to find the tangent at a given point of a curve.

Let us consider, for example, the parabola \( BDN \) with vertex \( D \) and of diameter \( DC \); let \( B \) be a point on it at which the line \( BE \) is to be drawn tangent to the parabola and intersecting the diameter at \( E \).

We choose on the segment \( BE \) a point \( O \) at which we draw the ordinate \( OI \); also we construct the ordinate \( BC \) of the point \( B \). We have then: \( CD/DI > BC^2/OI^2 \), since the point \( O \) is exterior to the parabola. But \( BC^2/OI^2 = CE^2/IE^2 \), in view of the similarity of the triangles. Hence \( CD/DI > CE^2/IE^2 \).

Now the point \( B \) is given, consequently the ordinate \( BC \), consequently the point \( C \), hence also \( CD \). Let \( CD = d \) be this given quantity. Put \( CE = a \) and \( CI = e \); we obtain

\[
\frac{d}{d - 3} > \frac{a^2}{a^2 + e^2 - 2ae}.
\]

Removing the fractions:

\[
da^2 + de^2 - 2dae > da^2 - a^2e.
\]

Let us then adequate, following the preceding method; by taking out the common terms we find:

\[
de^2 - 2dae \sim -a^2e,
\]

\(^6\)Isaac Newton: “When a quantity is greatest or least, at that moment its flow neither increases nor decreases: for if it increases, that proves that it was less and will at once be greater than it now is, and conversely if it decreases.” \textit{On the method of Series and Fluxions}, 1671.
or, which is the same, 
\[ de^2 + a^2e \sim 2dae. \]

Let us divide all terms by \( e \): On taking out \( de \), there remains \( a^2 = 2da \), consequently \( a = 2d \).

Thus we have proved that \( CE \) is the double of \( CD \) – which is the result.

This method never fails and could be extended to a number of beautiful problems; with its aid, we have found the centers of gravity of figures bound by straight lines or curves, as well as those of solids, and a number of other results which we may treat elsewhere if we have time to do so. ...

(2) CENTER OF GRAVITY OF PARABOLOID OF REVOLUTION, USING THE SAME METHOD

Let \( CBAV \) be a paraboloid of revolution, having or its axis \( IA \) and for its base a circle of diameter \( CIV \). Let us find its center of gravity by the same method which we applied for maxima and minima and for the tangents of curves; let us illustrate, with new examples and with new and brilliant applications of this method, how wrong those are who believe it may fail.

In order to carry out this analysis, we write \( IA = b \). Let \( O \) be the center of gravity, and \( a \) the unknown length of the segment \( AO \); we intersect the axis \( IA \) by any plane \( BN \) and put \( IN = e \), so that \( NA = b - e \).

It is clear that in this figure and in similar ones (parabolas or paraboloids) the centers of gravity of segments cut off by parallels to the base divide the axis in a constant proportion (indeed, the argument of Archimedes can be extended by similar reasoning from the case of a parabola to all parabolas and paraboloids of revolution[7]). Then the center of gravity of the segment of which \( NA \) is the axis and \( BN \) the radius of the base will divide \( AN \) in a point \( E \) such that \( NA/AE = IA/AO \), or, in formula, \( b/a = (b - e)/AE \).

The portion of the axis will then be \( AE = (ba - ae)/b \) and the interval between the two centers of gravity, \( OE = ae/b \).

Let \( M \) be the center of gravity of the remaining part \( CBRV \); it must necessarily fall between the points \( N \) and \( I \), inside the figure, in view of Archimedes’ postulate 9 in On the Equilibrium of planes[8], since \( CBRV \) is figure completely concave in the same direction.

But 
\[
\frac{\text{Part } CBRV}{\text{Part } BAR} = \frac{EO}{OM'},
\]

since \( O \) is the center of gravity of the total figure \( CAV \) and \( E \) and \( M \) are those of the parts.

Now in the paraboloid of Archimedes,
\[
\frac{\text{Part } CAV}{\text{Part } BAR} = \frac{IA^2}{NA^2} = \frac{b^2}{b^2 + e^2 - 2be};
\]

---

[7] “All parabolas” means “parabolas of higher order”, \( y = kx^n, n > 2 \)

[8] According to Heath: “In any figure whose perimeter is concave in (one and) the same direction the center of gravity must be within the figure.”
hence, by dividing,

\[
\frac{\text{Part } CBRV}{\text{Part } BAR} = \frac{2be - e^2}{b^2 + e^2 - 2be}.
\]

But we have proved that

\[
\frac{\text{Part } CBRV}{\text{Part } BAR} = \frac{EO}{OM}.
\]

Then in formulas,

\[
\frac{2be - e^2}{b^2 + e^2 - 2be} = \frac{EO(= ae/b)}{OM};
\]

hence

\[
OM = \frac{b^2ae + ae^3 - bae^2}{2b^2e - be^2}.
\]

From what has been established we see that the point \(M\) falls between points \(N\) and \(I\); thus \(OM < OI\); now, in formula, \(OI = b - a\). The question is then prepared from our method, and we may write

\[
b - a \sim \frac{b^2ae + ae^3 - bae^2}{2b^2e - be^2}.
\]

Multiplying both sides by the denominator and dividing by \(e\):

\[
2b^3 - 2b^2a - b^2e + bae \sim b^2a + ae^2 - 2bae.
\]

Since there are no common terms, let us take out all those in which \(e\) occurs let us equate the others:

\[
2b^3 2b^2a = b^2a, \text{ hence } 3a = 2b.
\]

Consequently,

\[
\frac{IA}{AO} = \frac{3}{2}, \text{ and } \frac{AO}{OI} = \frac{2}{1},
\]

and this was to be proved.

The same method applies to the centers of gravity of all parabolas ad infinitum as well as to those paraboloids of revolution. I do not have time to indicate, for example, how to look for the center of gravity in our paraboloid obtained by revolution about the ordinate\(^9\) it will be sufficient to say that, in this concoid, the center of gravity divides the axis into two segments in the ratio 11/5.

\(^9\)Here \(ACI\) is rotated about \(CI\).
2.2 Gottfried Leibniz: On his Discovery of Differential Calculus

I take for granted the following postulate\textsuperscript{10}:

*In any supposed transition, ending in any terminus, it is permissible to institute a general reasoning, in which the final terminus may also be included.*

For example, if A and B are any two quantities, of which the former is the greater and the latter is the less, and while B remains the same, it is supposed that A is continually diminished, until A becomes equal to B; then it will be permissible to include under a general reasoning the prior cases in which A was greater than B, and also the ultimate case in which the difference vanishes and A is equal to B. Similarly, if two bodies are in motion at the same time, and it is assumed that while the motion of B remains the same, the velocity of A is continually diminished until it vanishes altogether, or the speed of A becomes zero; it will be permissible to include this case with the case of the motion of B under one general reasoning. We do the same thing in geometry, when two straight lines are taken, produced in any manner, one VA being given in position or remaining in the same site, the other BP passing through a given point P, and varying in position while the point P remains fixed; at first indeed converging toward the line VA and meeting it in the point C; then, as the angle of inclination VGA is continually diminished, meeting VA in some more remote point (C), until at length from BP, through the position (B)P, it comes $\beta P$, in which the straight line no longer converges toward VA, but is parallel to it, and C is an impossible or imaginary point.

With this supposition it is permissible to include under some one general reasoning not only all the intermediate cases such as (B)P but also the ultimate case $\beta P$.

Hence also it comes to pass that we include as one case ellipses and the parabola, just as if A is considered to be one focus of an ellipse (of which V is the given vertex), and this focus remains fixed, while the other focus is variable as we pass from ellipse to ellipse, until at length (in the case when the line BP, by its intersection with the line VA, gives the variable focus) the focus C becomes evanescent\textsuperscript{11} or impossible, in which case the ellipse passes into a parabola. Hence it is permissible with our postulate that a parabola should be considered with ellipses under a common reasoning. Just as it is common practice to make use of this method in geometrical constructions, when they include under one general construction many different cases, noting that in a certain case the converging straight line passes into a parallel straight line, the angle between it and another straight line vanishing.

Moreover, from this postulate arise certain expressions which are generally used for the sake of convenience, but seem to contain an absurdity, although it is one that causes no


\textsuperscript{11}in the sense of “vanishing into the far distance”
hindrance, when its proper meaning is substituted. For instance, we speak of an imaginary point of intersection as if it were a real point, in the same manner as in algebra imaginary roots are considered as accepted numbers. Hence, preserving the analogy, we say that, when the straight line BP ultimately becomes parallel to the straight line VA, even then it converges toward it or makes an angle with it, only that the angle is then infinitely small; similarly, when a body ultimately comes to rest, it is still said to have a velocity, but one that is infinitely small; and, when one straight line is equal to another, it is said to be unequal to it, but that the difference is infinitely small; and that a parabola is the ultimate form of an ellipse, in which the second focus is at an infinite distance from the given focus nearest to the given vertex, or in which the ratio of PA to AC, or the angle BCA, is infinitely small.

Of course it is really true that things which are absolutely equal have a difference which is absolutely nothing; and that straight lines which are parallel never meet, since the distance between them is everywhere the same exactly; that a parabola is not an ellipse at all, and so on. Yet, a state of transition may be imagined, or one of evanescence, in which indeed there has not yet arisen exact equality or rest or parallelism, but in which it is passing into such a state, that the difference is less than any assignable quantity; also that in this state there will still remain some difference, some velocity, some angle, but in each case one that is infinitely small; and the distance of the point of intersection, or the variable focus, from the fixed focus will be infinitely great, and the parabola may be included under the heading of an ellipse (and also in the same manner and by the same reasoning under the heading of a hyperbola), seeing that those things that are found to be true about a parabola of this kind are in no way different, for any construction, from those which can be stated by treating the parabola rigorously.

Truly it is very likely that Archimedes, and one who seems so have surpassed him, Conon, found out their wonderfully elegant theorems by the help of such ideas; these theorems they completed with reductio ad absurdum proofs, by which they at the same time provided rigorous demonstrations and also concealed their methods. Descartes very appropriately remarked in one of his writings that Archimedes used as it were a kind of metaphysical reasoning (Caramuel would call it metageometry), the method being scarcely used by any of the ancients (except those who dealt with quadratrices); in our time Cavalieri has revived the method of Archimedes, and afforded an opportunity for others to advance still further. Indeed Descartes himself did so, since at one time he imagined a circle to be a regular polygon with an infinite number of sides, and used the same idea in treating the cycloid; and Huygens too, in his work on the pendulum, since he was accustomed to confirm his theorems by rigorous demonstrations; yet at other times, in order to avoid too great prolixity, he made use of infinitesimals ; as also quite lately did the renowned La Hire.

For the present, whether such a state of instantaneous transition from inequality to equality, from motion to rest, from convergence to parallelism, or anything of the sort, can be sustained in a rigorous or metaphysical sense, or whether infinite extensions successively greater and greater, or infinitely small ones successively less and less, are legitimate consid-
erations, is a matter that I own to be possibly open to question; but for him who would
discuss these matters, it is not necessary to fall back upon metaphysical controversies, such
as the composition of the continuum, or to make geometrical matters depend thereon. Of
course, there is no doubt that a line may be considered to be unlimited in any manner, and
that, if it is unlimited on one side only, there can be added to it something that is limited
on both sides. But whether a straight line of this kind is to be considered as one whole that
can be referred to computation, or whether it can be allocated among quantities which may
be used in reckoning, is quite another question that need not be discussed at this point.

It will be sufficient if, when we speak of infinitely great (or more strictly unlimited), or of
infinitely small quantities (i.e., the very least of those within our knowledge), it is understood
that we mean quantities that are indefinitely great or indefinitely small, i.e., as great as you
please, or as small as you please, so that the error that any one may assign may be less than
a certain assigned quantity. Also, since in general it will appear that, when any small error
is assigned, it can be shown that it should be less, it follows that the error is absolutely
nothing: an almost exactly similar kind of argument is used in different places by Euclid,
Theodosius and others; and this seemed to them to be a wonderful thing, although it could
not be denied that it was perfectly true that, from the very thing that was assumed as an
error, it could be inferred that the error was non-existent. Thus, by infinitely great and
infinitely small, we understand something indefinitely great, or something indefinitely small,
so that each conducts itself as a sort of class, and not merely as the last thing of a class. If
any one wishes to understand these as the ultimate things, or as truly infinite, it can be done,
and that too without falling back upon a controversy about the reality of extensions, or of
infinite continuums in general, or of the infinitely small, ay, even though he think that such
things are utterly impossible; it will be sufficient simply to make use of them as a tool that
has advantages for the purpose of the calculation, just as the algebraists retain imaginary
roots with great profit. For they contain a handy means of reckoning, as can manifestly be
verified in every case in a rigorous manner by the method already stated.

But it seems right to show this a little more clearly, in order that it may be confirmed
that the algorithm, as it is called, of our differential calculus, set forth by me in the year
1684, is quite reasonable. First of all, the sense in which the phrase “\(dy\) is the element of \(y\),”
is to be taken will best be understood by considering a line \(AY\) referred to a straight line
\(AX\) as axis.

Let the curve \(AY\) be a parabola, and let the tangent at the vertex \(A\) be taken
as the axis. If \(AX\) is called \(x\), and \(AY\), \(y\), and the latus–rectum is \(a\),
the equation to the parabola will be \(xx – ay\), and this holds good at
every point. Now, let \(A_1X = x\), and \(A_1Y = y\) and from the point
\(1Y\) let fall a perpendicular \(1YD\) to some greater ordinate \(2X_2Y\) that
follows, and let \(1Y_2X\), the difference between \(A_1X\) and \(A_2X\) be
called \(dx\); and similarly, let \(D_2Y\), the difference between \(1X_1Y\) and
\(2X_2Y\), be called \(dy\).
Then, since $y - xx : a$, by the same law, we have

$$y + dy = xx + 2x dx + dx dx : a;$$

and taking away the $y$ from the one side and the $xx : a$ from the other, we have left

$$dy : dx = 2x + dx : a;$$

and this is a general rule, expressing the ratio of the difference of the ordinates to the difference of the abscissae, or, if the chord $1Y_2Y$ is produced until it meets the axis in $T$, then the ratio of the ordinate $1Y$ to $T_1X$, the part of the axis intercepted between the point of intersection and the ordinate, will be as $2x + dx$ to $a$. Now, since by our postulate it is permissible to include under the one general reasoning the case also in which the ordinate $2X_2Y$ is moved up nearer and nearer to the fixed ordinate $1X_1Y$ until it ultimately coincides with it, it is evident that in this case $dx$ becomes equal to zero and should be neglected, and thus it is clear that, since in this case $T_1Y$ is the tangent, $1X_1Y$ is to $T_1X$ as $2x$ is to $a$.

Hence, it may be seen that there is no need in the whole of our differential calculus to say that those things are equal which have a difference that is infinitely small, but that those things can be taken as equal that have not any difference at all, provided that the calculation is supposed to be general, including both the cases in which there is a difference and in which the difference is zero; and provided that the difference is not assumed to be zero until the calculation is purged as far as is possible by legitimate omissions, and reduced to ratios of non-evanescent quantities, and we finally come to the point where we apply our result to the ultimate case.

Similarly, if $x^3 = aay$, then we have

$$x^3 + 3xx dx + 3x dx dx + dx dx dx = aay + aa dy$$

or cancelling from each side,

$$3xx dx + 3x dx dx + dx dx dx = aa dy$$

or

$$3xx + 3x dx + dx dx : aa = dy : dx = 1X_1Y : T_1X;$$

hence, when the difference vanishes, we have

$$3xx : aa = 1X_1Y : T_1X.$$
2.3 Isaac Newton: Quadrature of Curves

INTRODUCTION TO THE QUADRATURE OF CURVES

1. I consider mathematical quantities in this place not as consisting of very small parts; but as described by a continued motion. Lines are described, and thereby generated not by the apposition of parts, but by the continued motion of points; superficies by the motion of lines; solids by the motion of superficies; angles by the rotation of the sides; portions of time by a continual flux: and so in other quantities. These geneses really take place in the nature of things, and are daily seen in the motion of bodies. And after this manner the ancients, by drawing moveable right lines along inmoveable right lines, taught the genesis of rectangles.\cite{12}

2. Therefore considering that quantities, which increase in equal times, and by increasing are generated, become greater or less according to the greater or less velocity with which they increase and are generated; I sought a method of determining quantities from the velocities of the motions or increments, with which they are generated; and calling these velocities of the motions or increments *fluxions*, and the generated quantities *fluents*, I fell by degrees upon the method of fluxions, which I have made use of here in the quadrature of curves, in the years 1665 and 1666.

3. Fluxions are very nearly as the augments of the fluents generated in equal but very small particles of time, and, to speak accurately, they are in the *first ratio* of the nascent augments; but they may be expounded by any lines which are proportional to them.

4. Thus if the areas $ABC$, $ABDG$ be described by the ordinate $BC$, $BD$ moving along the base $AB$ with an uniform motion, the fluxions of these area’s shall be to one another as the describing ordinates $BC$ and $BD$, and may be expounded by these ordinates, because that these ordinates are as the nascent augments of the areas.

5. Let the ordinate $BC$ advance from it’s place into any new place $bc$. Complete the parallelogram $BCE$, and draw the right line $VTH$ touching the curve in $C$, and meeting the two lines $be$ and $BA$ produced in $T$ and $V$: and $Bc$, $Ec$ and $Cc$ will be the augments now generated of the absciss $AB$, the ordinate $BC$ and the curve line $ACc$; and the sides of the triangle $CET$ are in the *first ratio* of these augments considered as nascent, therefore the fluxions of $AB$, $BC$ and $AC$ are as the sides $CE$, $ET$ and $CT$ of that triangle $CET$, and may be expounded by these same sides, or, which is the same thing, by the sides of the triangle $VBC$, which is

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similar to the triangle \( CET \).

6. It comes to the same purpose to take the fluxions in the \textit{ultimate ratio} of the evanescent parts. Draw the right line \( Cc \), and produce it to \( K \). Let the ordinate \( bc \) return into its former place \( BC \), and when the points \( C \) and \( c \) coalesce, the right line \( CK \) will coincide with the tangent \( CH \), and the evanescent triangle \( CEc \) in its ultimate form will become similar to the triangle \( GET \), and its evanescent sides \( CE, Er \) and \( Cc \) will be ultimately among themselves as the sides \( CE, ET \) and \( CT \) of the other triangle \( GET \), are, and therefore the fluxions of the lines \( AB, BC \) and \( AC \) are in the same ratio. If the points \( C \) and \( c \) are distant from one another by any small distance, the right line \( CK \) will likewise be distant from the tangent \( CH \) by a small distance. That the right line \( CK \) may coincide with the tangent \( CH \), and the ultimate ratios of the lines \( CE, Ec \) and \( Cc \) may be found, the points \( C \) and \( c \) ought to coalesce and exactly coincide. The very smallest errors in mathematical matters are not to be neglected.

... 11. \textit{Let the quantity} \( x \) \textit{flow uniformly, and let it be proposed to find the fluxion of} \( x^n \).

In the same time that the quantity \( x \) by flowing, becomes \( x - o \), the quantity \( x^n \) will become \( (x + o)^n \), that is, by the method of infinite series, \( x^n + nox^{n-1} + \frac{n^2 - n}{2} oox^{n-1} + \&c. \) And the augments \( o \) and \( nox^{n-1} + \frac{n^2 - n}{2} oox^{n-1} + \&c \) are to one another as \( 1 \) and \( nx^{n-1} + \frac{n^2 - n}{2} ox^{n-1} + \&c \)

Now let these augments vanish, and their ultimate ratio will be \( 1 \) to \( nx^{n-1} \).

12. By like ways of reasoning, the fluxions of lines, whether right or curve in all cases, as likewise the fluxions of superficies angles and other quantities, may be collected by the method of \textit{prime} and \textit{ultimate} ratios. Now to institute an analysis after this manner in finite quantities and investigate the \textit{prime} or \textit{ultimate} ratios of these finite quantities when in their nascent or evanescent state, is consonant to the geometry of the ancients: and I was willing to show that, in the method of fluxions, there is no necessity of introducing figures infinitely small into geometry. Yet the analysis may be performed in any kind of figures, whether finite or infinitely small, which are imagined similar to the evanescent figures; as likewise in these figures, which, by the method of indivisibles, used to be reckoned as infinitely small, provided you proceed with due caution.

... 15. \textit{An equation being given involving any number of flowing quantities, to find the fluxions.}

\textit{Solution.} Let every term of the equation be multiplied by the index of the power of every flowing quantity that it involves, and in every multiplication change the side or root of the power into its fluxion, and the aggregate of all the products with their proper signs, will be
the new equation.

16. Explication. Let \( a, b, c, d, \&c. \) be determinate and invariable quantities, and let any equation be proposed involving the flowing quantities \( x, y, \&c. \) as \( x^3 - xy^2 + a^2z - b^3 = 0. \) Let the terms be first multiplied by the indexes of the power of \( x, \) and in every multiplication for the root, or \( x \) of one dimension write \( \dot{x}, \) and the sum of the factors will be \( 3\dot{x}x^2 - \dot{xy}^2. \) Do the same in \( y \) and there arises \(-2\dot{xy}\dot{y}.\) Do the same in \( z \), and there arises \( aa\ddot{z}. \) Let the sum of these products be put equal to nothing, and you’ll have the equation \( 3\dot{x}x^2 - \dot{xy}^2 - 2xy\dot{y} + aa\ddot{z} = 0. \) I say the relation of the fluxions is defined by this equation.

17. Demonstration. For let \( o \) be a very small quantity, and let \( o\dot{z}, o\dot{y}, o\dot{x} \) be the moments, that is the momentaneous synchronous increments of the quantities \( z, y, x. \) And if the flowing quantities are just now \( z, y, x, \) then after a moment of time, being increased by their increments \( o\dot{z}, o\dot{y}, o\dot{x}, \) these quantities shall become \( z + o\dot{z}, y + o\dot{y}, x + o\dot{x}: \) which being written in the first equation for \( z, y, x, \) give this equation \( x^3 + 3x^2o\dot{x} + 3xo\dot{x}\dot{x} + o^3\dot{x}^3 - xy^2 - o\dot{xy}^2 - 2o\dot{y}y^2 - 2\dot{x}^2\ddot{y} - x\dot{x}^2\ddot{y} - \dot{x}^3\dddot{y} + a^2z + a^2o\dot{z} - b^3 = 0. \)

Subtract the former equation from the latter, divide the remaining equation by \( o, \) and it will be \( o\dot{x}x^2 + 3x\dot{xo}x + x^3o^2 - \dot{xy}^2 - 2xy\ddot{y} - 2o\dot{y}y^2 - xo\dot{y}y - xo\dot{y}y - \dot{x}^2\ddot{y} + a^2\dot{x} = 0. \) Let the quantity \( o \) be diminished infinitely, and neglecting the terms which vanish, there will remain \( 3\dot{x}x - \dot{xy}^2 - 2xy\ddot{y} + a^2\ddot{z} = 0. \) Q.E.D.

18. A fuller explication. After the same manner if the equation were \( x^3 - xy^2 + aa\sqrt{ax - y^2 - b^3} = 0, \) thence would be produced \( 3x^2\dot{x} - \dot{xy}^2 - 2xy\ddot{y} + aa\sqrt{ax - y^2} = 0. \) Where if you would take away the fluxion \( \sqrt{ax - y^2}, \) put \( \sqrt{ax - y^2} = z, \) and it will be \( ax - y^2 = z^2, \) and by this proposition \( a\dot{x} - 2\dot{y}y = 2\ddot{z}, \) or \( \frac{a\dot{x} - 2\dot{y}y}{2z} = \ddot{z}, \) that is \( \frac{a\dot{x} - 2\dot{y}y}{2\sqrt{ax - y^2}} = \sqrt{ax - y^2}. \)

And thence \( 3x^2\dot{x} - \dot{xy}^2 = 2xy\ddot{y} + \frac{a^3\dot{x} - 2\dot{y}y}{2\sqrt{ax - y^2}} = 0. \)

19. And by repeating the operation, you proceed to second, third, and subsequent fluxions. Let \( zy^3 + z^4 + a^4 = 0 \) be an equation proposed, and by the first operation it becomes \( \dot{zy}^3 + 3\dot{z}y^2 + 4\ddot{z}z^3 = 0, \) by the second \( \ddot{z}y^3 + 6\dot{z}y^2y + 3\dot{z}y^2 + 6\dot{y}^2y - 4\ddot{z}z^3 - 12\dddot{z}z^2 = 0, \) by the third, \( \ddot{y}y^3 + 9\dot{y}y^2y + 18\dot{y}^2y + 3\dot{y}^2y + 18\ddot{y}yy + 6\dot{y}^2 - 4\ddot{z}z^3 + 36\dddot{z}zz^2 - 24\dddot{z}z^2 = 0. \)

20. But when one proceeds thus to second, third, and following fluxions, it is proper to consider some quantity as flowing uniformly, and for its first fluxion to write unity, for the second and subsequent ones, nothing. Let there be given the equation \( zy^3 - z^4 + a^4 = 0, \) as above; and let \( z \) flow uniformly, and let its fluxion be unity: then by the first operation it shall be \( y^3 + 3\dot{y}y^2 - 4z^3 = 0; \) by the second \( 6\dot{y}y^2 + 3\dot{y}^2y + 6\dot{y}^2 - 12z^2 = 0; \) by the third \( 9\dot{y}y^2 + 9\dot{y}^2y + 3\dot{y}^2y + 18\dot{y}yy + 6\dot{y}^2 - 4\ddot{z}z^3 + 36\dddot{z}zz^2 - 24\dddot{z}z^2 = 0. \)

But in equations of this kind it must be conceived that the fluxions in all the terms are of the same order, i.e., either all of the first order \( \dot{y}, \ddot{z}; \) or all of the second \( \dot{y}, \dot{y}^2, \ddot{z}, z^2; \) or all of the third \( \dot{y}, \dot{y}y, \dot{y}\ddot{z}, \dot{z}^2, \dddot{z}, \&c. \) And where the case is otherwise the order is to be completed by means of the fluxions of a quantity that flows uniformly, which fluxions
are understood. Thus the last equation, by completing the third order, becomes
\[ 9\dot{z}\ddot{y}y^2 + 18\dot{z}\dot{y}\ddot{y}yy + 18z\dot{y}\ddot{y}yy + 6z\ddot{y}y^3 - 24z\dot{z}^3 = 0. \]

## 2.4 George Berkeley: The Analyst

**THE ANALYST; OR, A DISCOURSE Addressed to an Infidel MATHEMATICIAN**[13]

Wherein It is examined whether the Object, Principles, and Inferences of the modern Analysis are more distinctly conceived, or more evidently deduced, than Religious Mysteries and Points of Faith.

I. Though I am a Stranger to your Person, yet I am not, Sir, a Stranger to the Reputation you have acquired, in that branch of Learning which hath been your peculiar Study; nor to the Authority that you therefore assume in things foreign to your Profession, nor to the Abuse that you, and too many more of the like Character, are known to make of such undue Authority, to the misleading of unwary Persons in matters of the highest Concernment, and whereof your mathematical Knowledge can by no means qualify you to be a competent Judge. Equity indeed and good Sense would incline one to disregard the Judgment of Men, in Points they have not considered or examined. But several who make the loudest Claim to those Qualities, do, nevertheless, the very thing they would seem to despise, clothing themselves in the Livery of other Men’s Opinions, and putting on a general deference for the Judgment of you, Gentlemen, who are presumed to be of all Men the greatest Masters of Reason, to be most conversant about distinct Ideas, and never to take things on trust, but always clearly to see your way, as Men whose constant Employment is the deducing Truth by the justest inference from the most evident Principles. With this bias on their Minds, they submit to your Decisions where you have no right to decide. And that this is one short way of making Infidels I am credibly informed.

II. Whereas then it is supposed, that you apprehend more distinctly, consider more closely, infer more justly, conclude more accurately than other Men, and that you are therefore less religious because more judicious, I shall claim the privilege of a Free-Thinker; and take the Liberty to inquire into the Object, Principles, and Method of Demonstration admitted by the Mathematicians of the present Age, with the same freedom that you presume to treat the Principles and Mysteries of Religion; to the end, that all Men may see what right you have to lead, or what Encouragement others have to follow you. It hath been an old remark that Geometry is an excellent Logic. And it must be owned, that when the Definitions are clear; when the Postulata cannot be refused, nor the Axioms denied; when from the distinct Contemplation and Comparison of Figures, their Properties are derived, by a perpetual well-connected chain of Consequences, the Objects being still kept in view, and the attention ever fixed upon them; there is acquired a habit of reasoning, close and exact and methodical: which habit strengthens and sharpens the Mind, and being transferred to other Subjects, is

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of general use in the inquiry after Truth. But how far this is the case of our Geometrical Analysts, it may be worth while to consider.

III. The Method of Fluxions is the general Key, by help whereof the modern Mathematicians unlock the secrets of Geometry, and consequently of Nature. And as it is that which hath enabled them so remarkably to outgo the Ancients in discovering Theorems and solving Problems, the exercise and application thereof is become the main, if not sole, employment of all those who in this Age pass for profound Geometers. But whether this Method be clear or obscure, consistent or repugnant, demonstrative or precarious, as I shall inquire with the utmost impartiality, so I submit my inquiry to your own Judgment, and that of every candid Reader. Lines are supposed to be generated by the motion of Points, Planes by the motion of Lines, and Solids by the motion of Planes. And whereas Quantities generated in equal times are greater or lesser, according to the greater or lesser Velocity, wherewith they increase and are generated, a Method hath been found to determine Quantities from the Velocities of their generating Motions. And such Velocities are called Fluxions: and the Quantities generated are called flowing Quantities. These Fluxions are said to be nearly as the Increments of the flowing Quantities, generated in the least equal Particles of time; and to be accurately in the first Proportion of the nascent, or in the last of the evanescent, Increments. Sometimes, instead of Velocities, the momentaneous Increments or Decrements of undetermined flowing Quantities are considered, under the Appellation of Moments.

IV. By Moments we are not to understand finite Particles. These are said not to be Moments, but Quantities generated from Moments, which last are only the nascent Principles of finite Quantities. It is said, that the minutest Errors are not to be neglected in Mathematics: that the Fluxions are Celerities, not proportional to the finite Increments though ever so small; but only to the Moments or nascent Increments, whereof the Proportion alone, and not the Magnitude, is considered. And of the aforesaid Fluxions there be other Fluxions, which Fluxions of Fluxions are called second Fluxions. And the Fluxions of these second Fluxions are called third Fluxions: and so on, fourth, fifth, sixth, &c. ad infinitum. Now as our Sense is strained and puzzled with the perception of Objects extremely minute, even so the Imagination, which Faculty derives from Sense, is very much strained and puzzled to frame clear Ideas of the least Particles of time, or the least Increments generated therein: and much more so to comprehend the Moments, or those Increments of the flowing Quantities in statu nascenti, in their very first origin or beginning to exist, before they become finite Particles. And it seems still more difficult, to conceive the abstracted Velocities of such nascent imperfect Entities. But the Velocities of the Velocities, the second, third, fourth, and fifth Velocities, &c. exceed, if I mistake not, all Humane Understanding. The further the Mind analyseth and pursueth these fugitive Ideas, the more it is lost and bewildered; the Objects, at first fleeting and minute, soon vanishing out of sight. Certainly in any Sense a second or third Fluxion seems an obscure Mystery. The incipient Celerity of an incipient Celerity, the nascent Augment of a nascent Augment, i.e. of a thing which hath no Magnitude: Take it in which light you please, the clear Conception of it will, if I mistake not, be
found impossible, whether it be so or no I appeal to the trial of every thinking Reader. And if a second Fluxion be inconceivable, what are we to think of third, fourth, fifth Fluxions, and so onward without end?

XXXV. ... And there is indeed reason to apprehend, that all Attempts for setting the abstruse and fine Geometry on a right Foundation, and avoiding the Doctrine of Velocities, Momentums, &c. will be found impracticable, till such time as the Object and the End of Geometry are better understood, than hitherto they seem to have been. The great Author of the Method of Fluxions felt this Difficulty, and therefore he gave in to those nice Abstractions and Geometrical Metaphysics, without which he saw nothing could be done on the received Principles; and what in the way of Demonstration he hath done with them the Reader will judge. It must, indeed, be acknowledged, that he used Fluxions, like the Scaffold of a building, as things to be laid aside or got rid of, as soon as finite Lines were found proportional to them. But then these finite Exponents are found by the help of Fluxions. Whatever therefore is got by such Exponents and Proportions is to be ascribed to Fluxions: which must therefore be previously understood. And what are these Fluxions? The Velocities of evanescent Increments? And what are these same evanescent Increments? They are neither finite Quantities nor Quantities infinitely small, nor yet nothing. May we not call them the Ghosts of departed Quantities?

2.5 Jean le Rond d’Alembert: *Differential Calculus*

What concerns us most here is the metaphysics of the *differential* calculus. This metaphysics, of which so much has been written, is even more important and perhaps more difficult to explain than the rules of this calculus themselves: various mathematicians, among them Role, who were unable to accept the assumption concerning infinitely small quantities, have rejected it entirely, and have held that the principle was false and capable of leading to error. Yet in view of the fact that all results obtained by means of ordinary Geometry can be established similarly and much more easily by means of the *differential* calculus, one cannot help concluding that, since this calculus yields reliable, simple, and exact methods, the principles on which it depends must also be simple and certain.¹⁴

Leibniz was embarrassed by the objections he felt to exist against infinitely small quantities, as they appear in the *differential* calculus; thus he preferred to reduce infinitely small to merely incomparable quantities. This, however, would ruin the geometric exactness of the calculations: is it possible, said Fontenelle, that the authority of the inventor would outweigh the invention itself? ...

Newton started out from another principle; and one can say that the metaphysics of this

great mathematician on the calculus of fluxions is very exact and illuminating, even though he allowed us only an imperfect glimpse of his thoughts.

He never considered the \textit{differential} calculus as the study of infinitely small quantities, but as the method of first and ultimate ratios, that is to say, the method of finding the limits of ratios. Thus this famous author has never differentiated quantities but only equations; in fact, every equation involves a relation between two variables and the differentiation of equations merely consists in finding the limit of the ratio of the finite differences of the two quantities contained in the equation. Let us illustrate this by an example which will yield the clearest idea as well as the most exact description of the method of the \textit{differential} calculus.

Let $AM$ be an ordinary parabola, the equation of which is $yy = ax$; here we assume that $AP = x$ and $PM = y$, and $a$ is a parameter. Let us draw the tangent $MQ$ to this parabola at the point $M$. Let us suppose that the problem is solved and let us take an ordinate $pm$ at any finite distance from $PM$; furthermore, let us draw the line $mMR$ through the points $M$, $m$. It is evident, first, that the ratio $MP/PQ$ of the ordinate to the subtangent is greater than the ratio $MP/PR$ or $mO/OM$ which is equal to it because of the similarity of the triangles $MOm$, $MPR$; second, that the closer the point $m$ is to the point $M$, the closer will be the point $R$ to the point $Q$, consequently the closer will be the ratio $MP/PR$ or $mO/OM$ to the ratio $MP/PQ$; finally, that the first of these ratios approaches the second one as closely as we please, since $PR$ may differ as little as we please from $PQ$. Therefore, the ratio $MP/PQ$ is the limit of the ratio of $mO$ to $OM$. Thus, if we are able to represent the ratio $mO/OM$ in algebraic form, then we shall have the algebraic expression of the ratio of $MP$ to $PQ$ and consequently the algebraic representation of the ratio of the ordinate to the subtangent, which will enable us to find this subtangent. Let now $MO = u$, $Om = z$; we shall have $ax = yy$ and $ax + au = yu + 2yz + zz$. Then in view of $ax = yy$ it follows that $zu = 2yx + zz$ and $z/u = a/(2y + z)$.

This value $a/(2y + z)$ is, therefore, in general the ratio of $mO$ to $OM$, wherever one may choose the point $m$. This ratio is always smaller than $a/2y$; but the smaller $z$ is, the greater the ratio will be and, since one may choose $z$ as small as one pleases, the ratio $a/(2y + z)$ can be brought as close to the ratio $a/2y$ as we like. Consequently $a/2y$ is the limit of the ratio $a/(2y + z)$, that is to say, of the ratio $mO/OM$. Hence $a/2y$ is equal to the ratio $MP/PQ$, which we have found to be also the limit of the ratio of $mO$ to $Om$, since two quantities that are limits of the same quantity are necessarily equal to each other. ...[proof of this statement]

From this it follows that $MP/PQ$ is equal to $a/2y$. Hence $PQ = 2yy/a = 2x$. Now, according to the method of the \textit{differential} calculus, the ratio of $MP$ to $PQ$ is equal to that of $dy$ to
\( dx \); and the equation \( ax = yy \) yields \( a \, dx = 2y \, dy \) and \( dy/dx = a/2y \). So \( dy/dx \) is the limit of the ratio of \( z \) to \( u \), and this limit is found by making \( z = 0 \) in the fraction \( a/(2y + z) \).

But, one may say, is it not necessary also to make \( z = 0 \) and \( u = 0 \) in the fraction \( z/y = a/(2y + z) \), which would yield \( 0/0 = a/2y? \) What does this mean? My answer is as follows. First, there is no absurdity involved; indeed \( 0/0 \) may be equal to any quantity one may wish: thus it may be \( = a/2y \). Secondly, although the limit of the ratio of \( z \) to \( u \) has been found when \( z = 0 \) and \( u = 0 \), because the latter one is not clearly defined; one does not know what is the ratio of two quantities that are both zero. This limit is the quantity to which the ratio \( z/u \) approaches more and more closely if we suppose \( z \) and \( u \) to be real and decreasing. Nothing is clearer than this; one may apply this idea to an infinity of other cases. ...

Following the method of differentiation (which opens the treatise on the quadrature of curves by the great mathematician Newton), instead of the equation \( ax + au = yy + 2yz + zz \) we might write \( ax + a\,0 = yy + 2y\,0 + 00 \), thus, so to speak, considering \( z \) and \( u \) equal to zero; this would have yielded \( 0/0 = a/2y \). What we have said above indicates both the advantage and the inconveniences of this notation: the advantage is that \( z \), being equal to 0, disappears without any other assumption from the ratio \( a/(2y + 0) \); the inconvenience is that the two terms of the ratio are supposed to be equal to zero, which at first glance does not present a very clear idea.

From all that has been said we see that the method of the differential calculus offers us exactly the same ratio that has been given by the preceding calculation. It will be the same with other more complicated examples. This should be sufficient to give beginners an understanding of the true metaphysics of the differential calculus. Once this is well understood, one will feel that the assumption made concerning infinitely small quantities serves only to abbreviate and simplify the reasoning; but that the differential calculus does not necessarily suppose the existence of those quantities; and that moreover this calculus merely consists in algebraically determining the limit of a ratio, for which we already have the expression in terms of lines, and in equating those two expressions. This will provide us with one of the lines we are looking for. This is perhaps the most precise and neatest possible definition of the differential calculus; but it can be understood only when one is well acquainted with this calculus, because often the true nature of a science can understood only by those that have studies this science.

### 2.6 Augustin–Louis Cauchy: Derivatives of a single variable

THIRD LESSON

*Derivatives of a single variable*

\[\text{\footnotesize \text{\cite{Cauchy}}.} \]

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\[\text{\footnotesize \cite{Cauchy}.} \]

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When the function \( y = f(x) \) remains continuous between two given limits of the variable \( x \), and one assigns to that variable a value contained between the two limits in question, an infinitely small increase given to the variable produces an infinitely small increase in the function itself. Consequently, if one then puts \( \Delta x = i \), the two terms of the ratio of differences
\[
\frac{\Delta y}{\Delta x} = \frac{f(x + i) - f(x)}{i}
\]
will be infinitely small quantities. But, while these two terms approach indefinitely and simultaneously to the limit zero, the ratio itself may converge towards another limit, perhaps positive, perhaps negative. This limit, when it exists, has a determined value for each particular value of \( x \); but it varies with \( x \). Thus, for example, if one takes \( f(x) = x^m \), where \( m \) denotes a whole number, the ratio between the infinitely small differences will be
\[
\frac{(x + i)^m - x^n}{i} = mx^{m-1} + \frac{m(m - 1)}{1.2}x^{m-2}i + \cdots + i^{m-1}
\]
and it will have for its limit the quantity \( mx^{m-1} \), that is to say, a new function of the variable \( x \). It will be the same in general; the form of the new function that serves as the limit of the ratio \( \frac{f(x + i) - f(x)}{i} \) will depend only on the form of the proposed function \( f(x) \). To indicate this dependence, one gives the new function the name derived function, and one denotes it, with the help of an accent, by the notation
\[
y' \text{ or } f'(x).
\]

In seeking derivatives of a single variable \( x \), it is useful to distinguish functions that one calls simple, and that one regards as the result of a single operation carried out on that variable, from functions that one constructs by means of several operations and which one calls compound. The simple functions produced by the operations of algebra and trigonometry (see the 1st part of Cours d’analyse, chapter I) may be reduced to the following
\[
a + x, \quad a - x, \quad ax, \quad \frac{a}{x}, \quad x^a, \quad A^x, \quad L(x), \quad \sin x, \quad \cos x, \quad \arcsin x, \quad \arccos x,
\]
where \( A \) denoted a constant number, \( a = \pm A \) a constant quantity, and the letter \( L \) indicating a logarithm taken in the system of which the base is \( A \). If one takes one of these simple function for \( y \), it will be easy in general to obtain the derived function \( y' \).

One will find, for example
\[
\text{for } y = a + x, \quad \frac{\Delta y}{\Delta x} = \frac{(a + x + i) - (a + x)}{i} = 1, \quad y' = 1;
\]
\[
\text{for } y = a - x, \quad \frac{\Delta y}{\Delta x} = \frac{(a - x + i) - (a - x)}{i} = 1, \quad y' = -1;
\]
for \( y = ax \), \( \frac{\Delta y}{\Delta x} = \frac{a(x + i) - ax}{i} = a \), \( y' = a \);

for \( y = \frac{a}{x} \), \( \frac{\Delta y}{\Delta x} = \frac{\frac{a}{x+1} - \frac{a}{x}}{i} = -\frac{a}{x(x+1)} \), \( y' = -\frac{a}{x^2} \).
3 Quadrature: How to find the area of shapes?

Proclus: I think it was in consequence of this problem [Euclid I.45: To construct a parallelogram equal to a given rectilinear figure in a given rectilinear angle] that the ancient geometers were led to investigate the squaring of the circle. For if a parallelogram is found equal to any rectilineal figure, it is worth inquiring whether it be not also possible to prove rectilineal figures equal to circular. Archimedes in fact proved that any circle is equal to a right-angled triangle wherein one of the sides about the right-angle is equal to the radius and the base to the perimeter.\(^{16}\)

3.1 Simon Stevin: Elements of the Art of Weighing

THEOREM II. PROPOSITION I\(^{17}\)

The center of gravity of any triangle is in the line drawn from the vertex to the middle point of the opposite side.

**Supposition.** Let \(ABC\) be a triangle of any form, in which from the angle \(A\) to \(D\), the middle point of the side \(BC\), there is drawn the line \(AD\).

**What is required to prove.** We have to prove that the center of gravity of the triangle is in the line \(AD\).

**Preliminary.** Let us draw \(EF\), \(GH\), \(IK\) parallel to \(BC\), intersecting \(AD\) in \(L\), \(M\), \(N\); after that \(EO\), \(GP\), \(IQ\), \(KR\), \(HS\), \(FT\), parallel to \(AD\).

**Proof.** Since \(EF\) is parallel to \(BC\), and \(EO\), \(FT\) to \(LD\), \(EFTO\) will be a parallelogram, in which \(EL\) is equal to \(LF\), also to \(OD\) and \(DT\), in consequence of which the center of gravity of the quadrilateral \(EFTO\) is in \(DL\), by the first proposition of this book.\(^{18}\) And for the same reason the center of gravity of the parallelogram \(GHSP\) will be in \(LM\), and of \(IKRQ\) in \(MN\); and consequently the center of gravity of the figure \(IKRHSFTOEPGQ\), composed of the aforesaid three quadrilaterals, will be in the line \(ND\) or \(AD\). Now as here three quadrilaterals have been inscribed in the triangle, so an infinite number of such quadrilaterals can be inscribed therein, and the center of gravity of the inscribed figure will always be (for the reasons mentioned

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\(^{18}\)Theorem I, Proposition I: “The geometrical center of any plane figure is also its center of gravity.”
above) in the line $AD$. But the more such quadrilaterals there are, the less the triangle $ABC$ will differ from the inscribed figure of the quadrilaterals. For if we draw lines parallel to $BC$ through the middle points of $AN$, $NM$, $ML$, $LD$, the difference of the last figure will be exactly half of the difference of the preceding figure.\textsuperscript{19} We can therefore, by infinite approximation, place within the triangle a figure such that the difference between the latter and the triangle shall be less than any given plane figure, however small. From which it follows that, taking $AD$ to be the center line of gravity,\textsuperscript{20} the apparent weight of the part $ADC$ will differ less from the apparent weight of the part $ADB$ than any plane figure that might be given, however small, from which I argue as follows.\textsuperscript{21}

A. Beside any different apparent gravities there may be placed a gravity less than their difference;

0. Beside the present apparent gravities $ADC$ and $ADB$ there cannot be placed any gravity less than their difference;

0. Therefore the present apparent gravities ADC and ADB do not differ.

Therefore $AD$ is the center line of gravity, and consequently the center of gravity of the triangle $ABC$ is in it.

Conclusion. The center of gravity of any triangle therefore is in the line drawn from the vertex to the middle point of the opposite side, which we had to prove.

Problem I, Proposition III. Given a triangle: to find its center of gravity.

Supposition. Let $ABC$ be a triangle.

What is required to find. We have to find its center of gravity.

Construction. There shall be drawn from $A$ to the middle point of $BC$ the line $AD$, likewise from $C$ to the middle point of $AB$ the line $CE$, intersecting $AD$ in $F$. I say that $F$ is the required center of gravity.

Proof. The center of gravity of the triangle $ABC$ is in the line $AD$, and also in $CE$, by the second proposition. It is therefore $F$, which we had to prove.

\textsuperscript{19} It is obviously assumed that the side $AB$ is divided into $n$ equal segments (in the figure $n = 4$). The difference between the area $\Delta$ of the triangle $ABC$ and that of the figure consisting of $(n-1)$ parallelograms is $\Delta/n$.

\textsuperscript{20} The statement that $AD$ is the center line of gravity seems to mean that $AD$ is the vertical through the point of suspension of the triangle at rest and hence, by the rule of statics quoted by Stevin earlier in the book (Book I, Prop. 6: The center of gravity of a hanging solid is always in its center line of gravity), the center of gravity is in $AD$.

\textsuperscript{21} Stevin here uses the form of the syllogism known in ancient logic as CAMESTRES (vowels AEE, A universal affirmation, as all P are Q, E universal negation, as no P are Q), ... The reasoning amounts to this: When we know that the difference of two quantities $A$ and $B$ is smaller than a quantity that can be taken as small as we like, then $A = B$. The reductio ad absurdum, typical of the Greeks, is replaced by a syllogism.
Conclusion. Given therefore a triangle, we have found its center of gravity, as required.

3.2 Bonaventura Cavalieri: *Geometria Indivisibilius Continuorum Nova Quadam Tatione Promota*

Book II, Definition I: If through opposite tangents to any given plane figure there are drawn two planes parallel to each other, either at right angles or inclined to the plane of the given figure, and produced indefinitely, one of which is moved towards the other, always remaining parallel until it coincides with it, then the single lines, which in the motion as a whole are the intersections of the moving plane and the given figure, collected together are called: All the lines of the figure, taken one of them as *regula*.  

... Book II, Theorem III: Plane figures have the same ratio to each other, as that of all their lines taken from whatever *regula*.

... Theorem IV. Proposition IV.

Suppose two plane figures, or solids, are constructed to the same altitude; moreover having taken straight lines in the planes, or planes in the solids, parallel to each other in whatever way, with respect to which the aforesaid altitude is taken, if it is found that segments of the taken lines intercepted in the plane figures, or portions of the taken planes intercepted in the solids, are proportional quantities, always in the same way in each figure, then the said figures will be to each other as any one of the former to the latter corresponding to it in the other figure.

First suppose the two plane figures constructed to the same altitude are CAM, CME, in which there are understood to be taken any two straight lines parallel to each to other, AE, BD, with respect to which the common altitude is understood to be taken; moreover let there be intercepted segments AM, BR, inside figure CAM, and ME, RD, inside figure CME; and suppose it is found that AM to ME is as BR to RD. I say that figure CAM to figure MCE, will be as AM to ME, or BR to RD, for since BD, AE, however taken, are parallel to each other, it is clear that any one of those which are said to be all the lines of figure CAM, taken from either AM, BR or *regula* to that which lies opposite to it in figure CME, will be as BR to RD, or as AM to ME; therefore as AM to ME, that is, as one of the former to one of the latter so will be all of the former, namely all the lines of figure CAM, from *regula AM*, to all the latter, that is, to all the lines of the figure CME, from *regula ME*. The indefinite number *n.* of all the former, or latter, is here the same for both, whatever it is.

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(for here the figures are to the same altitude, and to any supposed former line in the figure \( CAM \) there corresponds a latter constructed opposite to it in the other figure) so it cannot be but that all the lines of figure \( CAM \) are comparable to all the lines of figure \( CME \), since they have a ratio to them, as has been shown. And therefore all the lines of figure \( CAM \), from \( regula AM \), to all the lines of figure \( CME \), from \( regula ME \), will be as \( AM \) to \( ME \): but as all the lines of figure \( CAM \) to all the lines of figure \( CME \), so is figure \( CAM \) to figure \( CME \); therefore figure \( CAM \) to figure \( CME \) will be as \( BR \) to \( RD \), or \( AM \) to \( ME \), which it was required to show for plane figures.

But if we assume \( CAM, CM \), to be solid figures, and instead of the lines \( AM, BR, ME, RD \), we understand intercepted planes parallel to each other inside the figures \( CAM, CME \), and so constructed that planes \( AM, ME \), lie in the same plane, just as do the planes \( BR, RD \), with respect to which the aforesaid altitude is again understood to be taken, the proceeding by the same method we show that all the planes of figure \( CAM \) to all the planes of figure \( CME \), that is, the solid figure \( CAM \) to the solid figure \( CME \), are as the plane \( BR \) to the plane \( RD \), or as the plane \( AM \) to the plane \( ME \), which it was required to show also for solid figures.

### 3.3 Pierre de Fermat: Quadrature of Infinite Parabolas

**ON THE TRANSFORMATION AND EMENDATION OF EQUATIONS OF PLACE**

in order to compare curves in various ways with each other,

or with straight lines

**TO WHICH IS ADJOINED**

**THE USED OF GEOMETRIC PROGRESSIONS**

in the quadrature of infinite parabolas or hyperbolas.

Archimedes made use of geometric progressions only for the quadrature of one parabola. In the remaining comparisons of heterogeneous quantities he restricted himself merely to arithmetic progressions. Whether because he found geometric progressions less appropriate? Or because the required method with the particular progression used for squaring the first parabola could scarcely be extended to the others? I have certainly recognized, and proved, progressions of this kind very productive for quadratures, and my discovery, by which one may square both parabolas and hyperbolas by exactly the same method, I by no means unwillingly communicate to more modern geometers.

I attribute to geometric progressions only what is very well known, on which this whole method is based.

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23Pierre de Fermat (1601-1665), from *Varia Opera*, published 1679, likely written about 1658, based on discoveries in the 1640’s. Taken from *Mathematics Emerging, A Sourcebook 1540–1900*, Jacqueline Stedall, Oxford University Press, 2008, pages 78–84.
The theorem is this: Given any geometric progression whose terms decrease infinitely, as the difference of two [consecutive] terms constituting the progression is to the smaller of them, so is the greatest term of the progression to the rest taken infinitely.

This established, there is proposed first the quadrature of hyperbolas. Moreover we defined hyperbolas as infinite curves of various kinds, like DSEF, of which this is a property, that having placed at any given angle RAC its asymptotes, AR, AC, extended infinitely if one pleases but not cut by the curve, and taking whatever straight lines, GE, HI, ON, MP, RS, etc. parallel to one asymptote, we suppose that a certain power of the line AH to the same power of the line AG is as a power of the line GE, whether the same or different from the preceding one, to that same power of the line HI; moreover we understand the powers to be not only squares, cubes, square–squares, etc. of which the exponents are 2, 3, 4 etc. but also simple lines, whose power is one. I say, therefore, that all hyperbolas of this kind indefinitely, with one exception, which is that of Apollonius\(^24\) or the first, can be squared with the help of the same an always applicable method of geometric progressions.

Let there be, if one likes, a hyperbola of which it is the property that the square of the lines HA to the square of the lines AG is always as the line GE to the line HI, and that the square of OA to the square of AH is as the line HI to the line ON, et. I say that the infinite space whose base is GE, and with the curves ES for one side, but for the other the infinite asymptote GOR, is equal to a given rectilinear space. It is supposed that the terms of a geometric progression can be extended infinitely, of which the first is AG, the second AH, the third AO, etc., infinitely, and these approach each other by approximation as closely as is needed, so that by the method of Archimedes the parallelogram made by GE and GH adequately, as Diophantus says, to the irregular four–sided shape GHE, or very nearly equal.

\[GE \times GH\]

Likewise, the first of the straight line intervals of the progression GH, HO, OM, and so on, are similarly very nearly equal amongst themselves, so that we can conveniently use the method of exhaustion, and by Archimedean circumscriptions and inscriptions the ratio to be demonstrated can be established, which it is sufficient to have shown once, nor do I wish to repeat or insist more often on a method already sufficiently known to any geometer.

This said, since AH to AO is as AG to AH, so also will AO to AM be as AG to AH. So also will be the interval GH to HO, and the interval HO to HM, etc. Moreover the parallelogram made by EG and GH will be to the parallelogram made by HI and HO, as the parallelogram made by HI and HO to the parallelogram made by NO and OM, for the

\(^{24}\)The hyperbola \(xy = a^2\).
ratio of the parallelogram made by $GE$ and $GH$ to the parallelogram made by $HI$ and $HO$ is composed from the ratio of the line $GE$ to the line $HI$, and from the ratios of the line $GH$ to the line $HO$; and as $GH$ is to $HO$, so is $AG$ to $AH$, as we have shown. Therefore the ratio of the parallelogram made by $EG$ and $GH$ to the parallelogram made by $HI$ and $HO$ is composed from the ratio of the line $GE$ to the line $HI$, and from the ratios of the line $GH$ to the line $HO$; and as $GH$ is to $HO$, so is $AG$ to $AH$, as we have shown. Therefore the ratio of the parallelogram made by $EG$ and $GH$ to the parallelogram made by $HI$ and $HO$ is composed from the ratio of the line $GE$ to the line $HI$, and from the ratio $AG$ to $AH$, but as $GE$ is to $HI$ so by construction they are proportionals, as the line $AO$ to the line $GA$. Therefore the ratio of the parallelogram made by $EG$ and $GH$ to the parallelogram made by $HI$ and $HO$, will be composed of the ratios $AO$ to $GA$, and $AG$ to $AH$; but the ratio $AO$ to $AH$ is composed of these two. Therefore the parallelogram made by $GE$ and $GH$ is to the parallelogram made by $HI$ and $HO$, as $OA$ to $HA$; or as $HA$ to $AG$.

Similarly it can be proved that the parallelogram made by $HI$ and $HO$ is to the parallelogram made by $ON$ and $OM$, as $AO$ to $HA$, but the three lines that constitute the ratios of the parallelograms, namely $AO$, $HA$, $GA$, are proportionals by construction. Therefore the parallelograms made by $GE$ and $GH$, by $HI$ and $HO$, by $ON$ and $OM$, etc. taken indefinitely, will always be continued proportionals in the ratio of the lines $HA$ to $GA$. Therefore, from the theorem that is the foundation of this method, as $GH$, the difference of the terms of the progression, is to the smaller term $GA$, so will be the first term of the progression of the parallelograms, that is, the parallelogram made by $EG$ and $GH$, to the rest of the parallelograms taken infinitely, that is by the adequation of Archimedes, to the space contained by $HI$, the asymptote $HR$, and the curve $IND$ extended infinitely. But as $HG$ is to $GA$ so, taking as a common side the line $GE$, is the parallelogram made by $GE$ and $GH$ to the parallelogram made by $GE$ and $GA$. Therefore, as the parallelogram made by $GE$ and $GH$ is to that infinite figure whose base is $HI$, so is the same parallelogram made by $GE$ and $GH$ to the parallelogram made by $GE$ and $GA$; therefore the parallelogram made by $GE$ and $GA$, which is the given rectilinear space, adequates to the aforesaid figure. To which if there is added the parallelogram made by $GE$ and $GH$, which on account of the minute divisions vanishes and goes to nothing, there remains the truth, which may be easily confirmed by a more lengthy Archimedian demonstration, that the parallelogram $AE$ in this kind of hyperbola, is equal to the space contained between the base $GE$, the asymptote $GR$, and the curve $ED$, infinitely produced. Nor is it onerous to extend this discovery to all hyperbolas of this kind, except, as I said, one.

### 3.4 Evangelista Torricelli: On the Acute Hyperbolid Solid

Consider a hyperbola of which the asymptotes $AB$, $AC$ enclose a right angle. [Fig 1] If we rotate this figure about the axis $AB$, we create what we shall call an acute hyperbolic solid, which is infinitely long in the direction of $B$. Yet this solid is finite. It is clear that there are contained within this acute solid rectangles through the axis $AB$, such as $DEFG$. I claim
that such a rectangle is equal to the square of the semiaxis of the hyperbola.\footnote{Evangelista Torricelli (1608–1647), \textit{De solido hyperbolico acuto} c. 1643. Taken from \textit{A Source Book in Mathematics}, 1200-1800, edited by Dirk Jan Struik, Harvard University Press, Cambridge, 1969, pages 227–230.}

We draw from $A$, the center of the hyperbola, the semiaxis $AH$, which bisects the angle $BAC$. This gives us the rectangle $AIHC$, which is certainly a square (it is a rectangle and the angle at $A$ bisected by the axis $AH$.) Therefore the square of $AH$ is twice the square $AIHC$, or twice the rectangle $AF$, and therefore equal to the rectangle $DEFG$, as claimed.\footnote{$xy = \text{constant}$, from Apollonius Conics, Book II, prop 12.}

\begin{proof}

\textbf{Lemma 2.} All cylinders described within the acute hyperbolic solid and constructed about the common axis are isoperimetric (I always mean without their bases). Consider the acute solid with axis $AB$ [Fig 2] and visualize within it the arbitrary cylinders $CDEF$, $GHLI$, drawn about the common axis $AB$. The rectangles through the axes $CE, GL$ are equal and so the curved surfaces of the cylinders will be equal. Q.E.D.

\textbf{Lemma 3.} All isoperimetric cylinders (for instance, those that are drawn within the acute hyperbolic solid) are to each other as the diameters of their bases. Indeed, in Fig. 2, the rectangles $AE, AL$ are equal, hence $FE : IL = AI : AF$. The cylinder $CE$ has to cylinder $GL$ a ratio composed of $AF^2 : AI^2$ and $FE : IL$, or of $FA : IA$, or of $FA^2 : AI$ times $AF$. The cylinders $CE, GL$ are therefore to each other as $FA^2$ is to $AI$ times $AF$, and thus as line $FA$ is to line $AI$. Q.E.D.

\textbf{Lemma 4.} Let $ABC$ be an acute body with axis $BD$, $D$ the center of the hyperbola (where asymptotes meet), and $DF$ the axis of the hyperbola. We construct sphere $AEFC$ with center $D$ and radius $DF$. this is the largest sphere with center $D$ that can be described in the acute body. We take an arbitrary cylinder contained in the acute body, say $GIHL$. I claim that the surface of cylinder $GH$ is one–fourth that of the sphere $AEFC$.

Indeed, since the rectangle $GH$ through the axis of the cylinder is equal to $DF^2$, hence to the circle $AEFC$, there-
fore this cylindrical surface $GIHL = \frac{1}{4}$ the surface of the sphere $AEFC$, of which the great circle $AEFG$ is also one-fourth.

**Lemma 5.** The surface of any cylinder $GHIL$ described in the acute solid (the surface without bases) is equal to the circle of radius $DF$, which is the semiaxis, or half the latus versum of the hyperbola, for this is proved in the demonstratino of the preceeding lemma. 

**Theorem.** An acute hyperbolic solid, infinitely long, cut by a plane [perpendicular] to the axis, together with the cylinder of the same base, is equal to that right cylinder of which the base is the latus versum (that is, the axis) of the hyperbola, and of which the altitude is equal to the radius of the basis of this acute body.

Consider a hyperbola of which the asymptotes $AB$, $AC$ enclose a right angle. We draw from an arbitrary point $D$ of the hyperbola a line $DC$ parallel to $AB$, and $DP$ parallel to $AC$. Then the whole figure is rotated about $AB$ as an axis, so that the acute hyperbolic solid $EBD$ is formed together with a cylinder $FEDC$ with the same base. We extend $BA$ to $H$, so that $AH$ is equal to the entire axis, that is, the latus versum of the hyperbola. And on the diameter $AH$ we imagine a circle [in the plane] constructed perpendicularly to the asymptote $AC$, and over the base $AH$ we conceive a right cylinder $ACGH$ of altitude $AC$, which is the radius of the base of the acute solid. I claim that the whole body $FEBDC$, though long without end, yet is equal to the cylinder $ACGH$.

We select on the line $AC$ an arbitrary point $I$ and we form the cylindrical surface $ONLI$ inscribed in the acute solid about the axis $AB$, and likewise the circle $IM$ on the cylinder $ACGH$ parallel to the base $AH$. Then we have, according to our lemma: (cylindrical surface $ONLI$) is to (circle $IM$) as (rectangle $OL$ through the axis) is to (square of the radius of circle $OM$), hence as (rectangle $OL$) is to (square of the semiaxis of the hyperbola).

And this will always be true no matter where we take point $I$. Hence all cylindrical surfaces together, that is, the acute solid $EBD$ itself, plus the cylinder of the base $FEDC$, will be equal to all the circles together, that is, to the cylinder $ACGH$. Q.E.D.

### 3.5 Isaac Newton. Quadrature of Curves

Of Analysis by Equations

of an infinite Number of Terms.

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$27$ *Latus versum* is what we’d call the real axis. Thinking of $AB$ as the $y$-axis, the equation of the hyperbola is $xy = a^2/2$, when the length of the latus versum is $2a$. 

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1. The General Method, which I had derived some considerable Time ago, for measuring the Quantity of Curves, by Means of Series, infinite in the Number of Terms, is rather shortly explained, than accurately demonstrated in what follows.

2. Let the Base AB of any Curve AD have BD for its perpendicular Ordinate; and call AB = x, BD = y, and let a, b, c, &c. be given Quantities, and m and n whole Numbers. Then

The Quadrature of Simple Curves
RULE I.

3. If ax^{m/n} = y; it shall be \( \frac{an}{m+n}x^{m+n} = \text{Area ABD} \).

The thing will be evident by an Example.

1. If \( x^2(= 1x^\frac{2}{1}) = y \), that is \( a = 1 = n \), and \( m = 2 \), it shall be \( 1 \cdot x^3 = ABD \).
2. Suppose \( 4\sqrt{x}(= 4x^{\frac{1}{2}}) = y \); it will be \( 3 \cdot x^{\frac{3}{2}} = ABD \).
3. If \( \sqrt[3]{x^5}(= x^\frac{3}{5}) = y \); it will be \( \frac{3}{5} \cdot x^\frac{8}{5} (= \frac{3}{5} \sqrt{x^8}) = ABD \).
4. If \( \frac{1}{x^7}(= x^{-2}) = y \); that is if \( a = 1 = n \), and \( m = -2 \); It will be \( \frac{1}{-1} \cdot x^\frac{-1}{7} = -x^{-1}(= \frac{-1}{x}) = \alpha BD \), infinitely extended towards \( \alpha \), which the Calculation places negative, because it lies upon the other side of the line BD.
5. If \( \frac{1}{\sqrt{x}} (= x^{\frac{-1}{2}}) = y \); it will be \( \frac{2}{\sqrt{x}}(= \frac{2}{x^{\frac{1}{2}}}) = \alpha BD \).
6. If \( \frac{1}{x}(= x^{-1}) = y \); it will be \( \frac{1}{0} \cdot x^0 = \frac{1}{0} \cdot 1 = \frac{1}{0} = \) an infinite Quantity; such as is the Area of the Hyperbola upon both Sides of the Line BD.

The Quadrature of Curves composed of simple ones.
RULE II.

4. If the Value of \( y \) be made up of several such Terms, the Area likewise shall be made up of the Areas which result from every one of the Terms.

The first Examples.
5. If it be \( x^2 + x^{\frac{3}{2}} = y \); it will be \( \frac{1}{3}x^3 + \frac{2}{5}x^{\frac{5}{2}} = ABD \).

For if it be always \( x^2 = BF \) and \( x^{\frac{3}{2}} = FD \), you will have by the preceding Rule \( \frac{1}{3}x^3 = \text{Superficies } AFB \); described by the line \( BF \); and \( \frac{2}{5}x^{\frac{5}{2}} = AFD \) described by \( DF \); wherefore \( \frac{1}{3}x^3 + \frac{2}{5}x^{\frac{5}{2}} = \) the whole area \( ABD \).

Thus if it be \( x^2 - x^{\frac{1}{2}} = y \); it will be \( \frac{1}{3}x^3 - \frac{2}{5}x^{\frac{5}{2}} = ABD \). And if it be \( 3x - 2x^2 + x^3 - 5x^4 = y \); it will be \( \frac{3}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4 - x^5 = ABD \).

The second Examples.

6. If \( x^{-2} + x^{-\frac{3}{2}} = y \); it will be \( -x^{-1} - 2x^{-\frac{1}{2}} = \alpha BD \). Or if it be \( x^{-2} - x^{-\frac{3}{2}} = y \); it will be \( x^{-1} + 2x^{-\frac{1}{2}} = \alpha BD \).

And if you change the Signs of the Quantities, you will have the affirmative Value \( (x^{-1} + 2x^{-\frac{1}{2}}, \text{ or } x^{-1} - 2x^{-\frac{1}{2}}) \) of the Superficies \( \alpha BD \), provided the whole of it fall above the Base \( AB\alpha \).

7. But if any Part fall below (which happens when the Curve descussates or crosses it’s Base betwixt \( B \) and \( \alpha \) as you see here in \( \delta \)) you are to subtract that Part from the Part above the Base; and so you shall have the Value of the Difference: but if you would have their Sum, seek both the Superficie’s separately, and add them. And the same thing I would have observed in the other Examples belonging to this Rule.

...
The Demonstration

55. Or universally, if \( \frac{n}{m+n} \times ax^{\frac{m+n}{n}} = z \); or, putting \( \frac{na}{m+n} = c \) and \( m+n = p \), if \( cx^{\frac{p}{n}} = z \); or \( c^nx^p = z^n \): Then by substituting \( x + o \) for \( x \), and \( z + ov \) (or which is the same \( z + oy \)) for \( z \), there arises \( c^n \int x^p + pox^{p-1}, \&c. = z^n + noyz^{n-1} \&c. \) the other Terms, which would at length vanish being neglected. Now taking away \( c^nx^p \) and \( z^n \) which are equal, and dividing the Remainders by \( o \), there remains \( c^np x^{p-1} = nyz^{n-1} = \frac{nyz^n}{z} = \frac{yc^n x^p}{cx^{\frac{p}{n}}} \), or, by dividing by \( c^nx^p \), it shall be \( px^{p-1} = \frac{ny}{cx^{\frac{p}{n}}} \); or \( pce^{\frac{n-n}{m+n}} = ny \); or by restoring \( \frac{na}{m+n} \) for \( c \), and \( m+n \) for \( p \), that is \( m \) for \( p - n \), and \( na \) for \( pc \), it becomes \( ax^{\frac{m}{n}} = y \). Wherefore conversely, if \( ax^{\frac{m}{n}} = y \), it shall be \( \frac{n}{m+n} ax^{\frac{m+n}{n}} = z \). Q.E.D.

3.6 John Wallis: On Indivisibles

Proposition 1. Plane figures considered according to the method of indivisibles.

I suppose at the start (according to the Geometria indivisibilium of Bonaventura Cavalieri) that any plane whatever consists, as it were, of an infinite number of parallel lines. Or rather (which I prefer) of an infinite number of parallelograms of equal altitude; of which indeed the altitude of a single one is \( \frac{1}{\infty} \) of the whole altitude, or an infinitely small divisor; (for let \( \infty \) denote an infinite numbers); and therefore the altitude of all of them at once is equal to the altitude of the figure.\(^{29}\)

Moreover, in whichever way the things is explained (whether by infinitely many parallel lines, or by infinitely many parallelograms of equal altitude interposed between those infinitely many lines) it comes down to the same thing. For a parallelogram whose altitude is supposed infinitely small, that is, nothing (for an infinitely small quantity is just the same as no quantity) is scarcely other than a line. (In this at least they differ, that a line is here supposed to be dilatable, or at least to have a certain such thickness that by infinite multiplication it can acquire a definite altitude or latitude, namely as much as the altitude of the figure.) Therefore from now on (partly because strictly speaking this seems to have been the case in Cavalieri’s method of indivisibles, partly also so that we may deliberate with brevity) we sometimes call those infinitely tiny parts of figures (or, of infinitely tiny altitude) by the name of LINES rather than PARALLELOGRAMS, at least when we do not have to consider the determination of the altitude. Moreover, when we do have to take into consideration the determination of the altitude (as sometimes happens) those tiny altitudes must have a ratio, so that infinitely multiplied they are assumed to become equal to the whole altitude of the figure.

\(^{29}\)meaning, multiplied by

3.7 Thomas Hobbes: on Wallis’ Indivisibles

Therefore, though your Lemma be true, and by me (Chap.13, Art.5) demonstrated; yet you did not know why it is true; which also appears most evidently in the first Proposition of your Conique-sections. Where first you have this, That a Parallelogram whose Altitude is infinitely little, that is to say, none, is scarce anything else but a Line. Is this the Language of Geometry? How do you determine this word scarce? The least Altitude, is Somewhat or Nothing. If Somewhat, then the first character of your Arithmetical Progression must not be zero; and consequently the first eighteen Propositions of this your Arithmetica Infinitorum are all naught. If Nothing, then your whole figure is without Altitude, and consequently your Understanding naught. Again, in the same Proposition, you say thus, We will sometimes call those Parallelograms rather by the name of Lines then of Parallelograms, at least when there is no consideration of a determinate Altitude; But where there is a consideration of a determinate Altitude (which will happen sometimes) there that little Altitude shall be so far considered, as that being infinitely multiplied it may be equall to the Altitude of the whole Figure. See here in what a confusion you are when you resist the truth. When you consider no determinate Altitude (that is, no Quantity of Altitude) then you say your Parallelogram shall be called a Line. But when the Altitude is determined (that is, when it is Quantity) then you will call it a Parallelogram. Is not this the very same doctrine which you so much wonder at and reprehend in me, in your objection to my eighth Chapter, and your word considered used as I use it? 'Tis very ugly in one that so bitterly reprehendeth a doctrine in another, to be driven upon the same himself by the force of truth when he thinks not on’t. Again, seeing you admit in any case, the infinitely little altitudes to be quantity, what need you this limitation of yours, so far forth as that by multiplication they may be made equall to the Altitude of the whole figure? May not the half, the third, the fourth, or the fifth part, etc. be made equall to the whole by multiplication? Why could you not have said plainly, so far forth as that every one of those infinitely little Altitudes be not only something but an aliquot part [divisor] of the whole? So you will have an infinitely little Altitude, that is to say a Point, to be both nothing and something and an aliquot part. And all this proceeds from not understanding the grounds of your Profession.

3.8 Leonhard Euler: From Foundations of Integral Calculus

Definition 1

1. Integral calculus is the method of finding, from a given relationship between differentials, a relationship between the quantities themselves: and the operation by which this is

Corollary 1

2. Therefore where differential calculus teaches us to investigate the relationship between differentials from a given relationship between variable quantities, integral calculus supplies us with the inverse method.

Corollary 2

3. Clearly just as in Analysis two operations are always contrary to each others, as subtraction to addition, division to multiplication, extraction of roots to raising of power, so also by similar reasoning integral calculus is contrary to differential calculus.

Corollary 3

4. Given any relationship between two variable quantities $x$ and $y$, in differential calculus there is taught a method of investigating the ratio of the differentials $dx : dy$; but if from this ratio of differentials there can in turn be determined the relationship of the quantities $x$ and $y$, this matter is assigned to integral calculus.\footnote{Euler used the notation $\partial x : \partial y$ where we now use $dx : dy$. His notation has been replaced by the modern throughout. JS}

Commentary 1

5. I have already noted that in differential calculus the question of differentials must be understood not absolutely but relatively, thus if $y$ is any function of $x$, it is not the differential $dy$ itself but the ratio to the differential $dx$ that is determined. For since all differentials in themselves are equal to nothing, whatever may be the function of $y$ of $x$, always $dy = 0$, and thus it is not possible to search more generally for anything absolute. But the question must be rightly proposed thus, that while $x$ takes an infinitely small and therefore vanishing increment $dx$, there is defined a ratio of the increment of the function $y$, which it takes as a result, to $dx$; for although both are $= 0$, nevertheless there stands a definite ratio between them, which is correctly investigated by differential calculus. Thus if $y = xx$, it is shown by differential calculus that $\frac{dy}{dx} = 2x$, nor is this ratio of increments true unless the increment $dx$, from which $dy$ arises, is put equal to nothing. But nevertheless, having observed this truth about differentials, one can tolerate common language, in which differentials are spoken of as absolutes, which always at least in the mind referring to the truth. Properly, therefore, we say if $y = xx$ then $dy = 2x \, dx$, even though it would not be false if anyone said $dy = 3x \, dx$ or $dy = 4x \, dx$, for since $dx = 0$ and $dy = 0$, these equalities would equally well stand; but only the first of the ratios, $\frac{dy}{dx} = 2x$, is agreed to be true.
Commentary 2

6. In the same way that the differential calculus is called by the English the method of fluxions, so integral calculus is usually called by them the inverse method of fluxions, since indeed one reverts from fluxions to fluent quantities. For what we call variable quantities, the English more fitly call by the name of fluent quantities, and their infinitely small or vanishing increments they call fluxions, so that fluxions are the same to them as differentials to us. This variation in language is already established in use, so that a reconciliation is scarcely ever to be expected; indeed we imitate the English freely in forms of speech, but the notation that we use seems to have been established a long time before their notation. And indeed since so many books are already published written either way, a reconciliation of this kind would be of no use.

Definition 2

7. Since the differentiation of any function of \( x \) has a form of this kind \( X \, dx \), when such a differential form \( X \, dx \) is proposed, in which \( X \) is any function of \( x \), that function whose differential \( = X \, dx \) is called its integral, and is usually indicated by the prefix \( \int \), so that \( \int X \, dx \) denotes that variable quantity whose differential \( = X \, dx \).

Corollary 1

8. Therefore from the integral of the proposed differential \( X \, dx \), or from the function of \( x \) whose differential \( = X \, dx \), both of which will be indicated by this notation \( \int X \, dx \), there is to be investigated whatever is to be explained by integral calculus.

Corollary 2

9. Therefore just as the letter \( d \) is the sign of differentiation, so we use the letter \( \int \) as the sign of integration, and thus these two signs are mutually contrary to each other, as though they destroy each other: certainly \( \int dX = X \), because the former is denoted by the quantity whose differential is \( dX \), which in both cases is \( X \).

Corollary 3

10. Therefore since the differentials of these functions of \( x \)

\[
x^2, x^n, \sqrt{(aa - xx)}
\]

are

\[
2x \, dx, nx^{n-1} \, dx, \frac{-x \, dx}{\sqrt{(aa - xx)}}
\]
then adjoining the sign of integration \( \int \), they are seen to become:

\[
\int 2x \, dx = xx; \int nx^{n-1} \, dx = x^n, \int \frac{-x \, dx}{\sqrt{(aa - xx)}} = \sqrt{(aa - xx)}
\]
whence the use of this sign is more clearly seen.
3.9 Bernhard Riemann. On the Representation of a Function

On the concept of a definite integral and the extent of its validity. 34

The uncertainty that still prevails on some fundamental points of the theory of definite integrals requires us to begin with something about the concept of a definite integral and the extent of its validity.

So, first: What is one to understand by \( \int_{a}^{b} f(x) \, dx \)?

To establish this, we take a sequence of values \( x_1, x_2, \ldots, x_{n-1} \), following one after another between \( a \) and \( b \) in order of size, and for the sake of brevity we denote \( x_1 - a \) by \( \delta_1 \), \( x_2 - x_1 \) by \( \delta_2 \), \ldots, \( b - x_{n-1} \) by \( \delta_n \), and by \( \varepsilon \) a proper fraction. Then the value of the sum

\[
S = \delta_1 f(a + \varepsilon_1 \delta_1) + \delta_2 f(x_1 + \varepsilon_2 \delta_2) + \delta_3 f(x_2 + \varepsilon_3 \delta_3) + \cdots + \delta_n f(x_{n-1} + \varepsilon_n \delta_n)
\]

will depend on the choice of the intervals \( \delta \) and the quantities \( \varepsilon \). If this now has the property that, however \( \delta \) and \( \varepsilon \) are chosen, it comes infinitely close to a fixed limit \( A \) when all the \( \delta \) become infinitely small, then this value is called by \( \int_{a}^{b} f(x) \, dx \).

If it does not have this property, then \( \int_{a}^{b} f(x) \, dx \) has no meaning. But even then, there have been several attempts to attribute a meaning to this symbol, and among these extensions of the concept of a definite integral there is one accepted by all mathematicians. Namely, if the function \( f(x) \) become infinitely large when the variable approaches a particular value \( c \) in the interval \((a, b)\), then clearly the sum \( S \), no matter what order of smallness one ascribes to \( \delta \), can take any arbitrary value; thus it has no limiting value, and \( \int_{a}^{b} f(x) \, dx \) according to the above would have no meaning. But if then

\[
\int_{a}^{c-\alpha_1} f(x) \, dx + \int_{c+\alpha_2}^{b} f(x) \, dx
\]

approaches a fixed limit, as \( \alpha_1 \) and \( \alpha_2 \) become infinitely small, then one understands by this limit \( \int_{a}^{b} f(x) \, dx \).

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