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Chapter 1

Overview. Initial Questions

1.1 Quadrature: How to find the area of shapes?

Proclus, *A Commentary on the First Book of Euclid’s Elements, c. 450:* “I think it was in consequence of this problem that the ancient geometers were led to investigate the squaring of the circle. For if a parallelogram is found equal to any rectilineal figure, it is worth inquiring whether it be not also possible to prove rectilineal figures equal to circular. Archimedes in fact proved that any circle is equal to a right-angled triangle wherein one of the sides about the right-angle is equal to the radius and the base to the perimeter.”

Euclid knows the area of rectangular shapes (via I.45). He can compare the areas of triangles and parallelograms (via VI.1). He can relate the area of circles to their diameters (via XII.2). What about the area of (sections) of other shapes, such as parabolas and ellipses?

Perhaps use Archimedes on the area of a circle as the initial reading.

1.2 Tangents: How to find tangent lines?

By Euclid Definition III.1 a tangent line is one that touches a curve but does not cut it. Euclid III.16 shows how to find the tangent line to circle. Apollonius I.33, 34 find the tangents to conics. What about other curves? [It would be helpful to have read part of Book 2 of Descartes, as he introduces a bunch more curves.]

Is there a hunk of Archimedes On Spirals that would be usable?

---

1 Euclid I.45: To construct a parallelogram equal to a given rectilinear figure in a given rectilinear angle.
2 [Th vol 2, 317]
1.3 Infinity, Infinitesimals, Continuity and Completeness.

In Euclid III.16 we meet the so-called “horn angles;” angles that are both non-zero and smaller than any rational angle. Are there really such numbers – positive, infinitely small but still non-zero? What sorts (if any) of infinity large and infinitely small (positive) integers are allowed? What can we do with these numbers?

Euclid definitions I.1 and I.2 cap lines with points. Are lines simply a collection of points? If so, how many? In X.1 we learn that lines (and magnitudes, in general) are infinitely divisible. This allows us to cut a line in half, discard one of those halves, and repeat this process indefinitely, without ever exhausting the line. What does this mean?

Aristotle: “motion is supposed to belong to the class of things which are continuous; and the infinite presents itself first in the continuous – that is how it comes about that ‘infinite’ is often used in definitions of the continuous (‘what is infinitely divisible is continuous’)

Aristotle instructs us “to discuss the infinite and to inquire whether there is such a thing or not, and, if there is, what is it.”

---

3 Physics Book II, Chapter 1. [ref 4.95]
4 Physics, Book III, chapter 4. [ref 4.95]
Chapter 2

Hippocrates of Chios, and his Lunes

2.1 quotes

Simplicius, *In Aristotelis Physica*: “Eudemus, however, in his History of Geometry says that Hippocrates did not demonstrate the quadrature of the lune on the side of a square but generally, as one might say. For every lune has an outer circumference equal to a semicircle or greater or less, and if Hippocrates squared the lune having an outer circumference equal to a semicircle and greater and less, the quadrature would appear to be proved generally. I shall set out what Eudemus wrote word for word, adding only for the sake of clearness a few things taken from Euclid’s Elements on account of the summary style of Eudemus, who set out his proofs in abridged form in conformity with the ancient practice. He writes thus in the second book of the History of Geometry:

“The quadratures of lunes, which seemed to belong to an uncommon class of propositions by reason of the close relationship to the circle, were first investigated by Hippocrates, and seemed to be set out in correct form; therefore we shall deal with them at length and go through them. He made his starting-point, and set out as the first of the theorems useful to this purpose, that similar segments of circles have the same ratios as the squares on their bases. And this he proved by showing that the squares on the diameters have the same ratios as the circles. Having first shown this he described in what way it was possible to square a lune whose outer circumference was a semicircle. He did this by circumscribing about a right-angled isosceles triangle a semicircle and about the base a segment of a circle similar to those cut off by the sides. Since the segment about the base is equal to the sum of those about the sides, it follows that when the part of the triangle above the segment about the base is added to both, the lune will be equal to the triangle.”

[ref 2.5]
Joannes Philoponus, *In Aristotelis Physica*, c. 550: “Hippocrates of Chios was a merchant who fell in with a pirate ship and lost all his possessions. He came to Athens to prosecute the pirates and, staying a long time in Athens by reason of the indictment, consorted with philosophers, and reached such proficiency in geometry that he tried to affect the quadrature of the circle. He did not discover this, but having squared the lune he falsely thought from this that he could square the circle also. For he thought that from the quadrature of the lune the quadrature of the circle could also be calculated.\(^2\)

### 2.2 Proof

Hippocrates Quadrature of the Lunules, According to Simplicus\(^3\)

The squaring of the lunules, considered as remarkable figures on account of their connection with the circle, was first formulated by Hippocrates and his explanation was considered to be in good order. Let us therefore attack the matter and study it.

He considered as the foundation as the first of the propositions which serves his purpose, that similar segments of circles are in the same ratio as the squares of their bases. He demonstrated this by showing first that the squares of the diameters have the same ratio as the circles. For the ratio of the circles is the same as that of similar segments, since similar segments are segments which form the same part of the circle.

After having proved this, he raised first of all the question how to square a lunule whose exterior boundary is a semicircle. He accomplished this by circumscribing a semicircle about an isosceles right triangle, and by constructing on the base a circular segment similar to the segments cut off by the right sides. Because the segment on the base is equal to (the sum of) the two (segments) on the other (sides), it follows, when the part of the triangle, which lies above the segment on the base, is added to both, that the lunule is equal to the triangle. Since it has now been shown that the lunule is equal to the triangle, it can be squared. Thus, by taking a semicircle as the external boundary of the lunule, he could readily square the lunule.

Next he started from an (external boundary) greater than a semicircle. He constructed a trapezoid, of which three sides were equal to each other, while the fourth side, the longer of the two parallel sides, had a square equal to three times that of any of the others. He circumscribed a circle about the trapezoid,

\(^2\) [ref 2.5]

\(^3\) [vdW, pages 132–135.] Figures also copied from vdW. According to vdW, “The manner in which [Hippocrates] squares such lunules can be learned from a famous fragment, copied word for word by Simplicius, according to his own statement, from the History of mathematics of Eudemus, with the addition of a few clarifying referenced to Euclid.” This text was then “purified”, according to vdW, by a number of modern authors, into that given here.
and constructed on the largest side a segment, similar to the segments which each of the other three sides cut from the circle.

Van der Waerden: "Eudemus leaves the squaring of the lunules to the reader. The lunule obviously equals the area of the trapezoid; the proof is similar to that of the previous case."

By drawing the diagonal of the trapezoid one sees that the segment in question is greater than a semicircle. For the line which subtends two sides of the trapezoid must of necessity have a square which is more than twice as great as that on the other remaining (side), while the squared longest side of the trapezoid is less than the diagonal and that one of the other sides which, with the diagonal, subtends this longest side. But then the angle which stands on the longest side is acute. Therefore the segment in which it is inscribed exceeds a semicircle; and this is the external boundary of the lunule.

But supposing that it were less than a semicircle, he proved it on the basis of a construction like the following. Let there be a circle of diameter $AB$ and center $K$, and let $\Gamma \Delta$ bisect the (line) $BK$ perpendicularly. And let $EZ$ lie between this (perpendicular bisector) and the circle, directed towards $B$, while its square is one-and-one half times as great as that of the radius.

Now let $EH$ be drawn parallel to $AB$ and let $K$ be joined to the points $E$ and $Z$. And let the line towards $Z$ meet the line $EH$ in $H$, and let the lines joining $B$ to $Z$ and to $H$ also be drawn. Then it is clear that the extension of $BZ$ will pass through $E$ and that $BH$ will be equal to $EK$. If this is so, then a circle will circumscribe the trapezoid $EKBH$. And the trapezoid will have in its interior a circular segment circumscribed about the triangle $EZH$. And the lunule that results will be equal in area to the rectilinear figure, composed of the three triangles ($EZK, HZB$ and $KZB$). For, the segments which $EZ$ and $ZH$ cut off from the rectilinear figure inside the lunule are equal to the segments outside the rectilinear figure, since each of the two on the inside is $3/2$ times as great as the outside ones. If, therefore, the lunule consists of the three segments and the rectilinear figure, except for the two segments, and if the rectilinear figure is itself obtained by adding the two segments but removing the three, then, the two segments being equal to the three, the lunule will be equal to the rectilinear figure.

He proves that the outer boundary of this lunule is less than a semicircle, by showing that the angle inscribed in the outer segment is obtuse. And that this angle is obtuse, he proves as follows: Since the square on the line $EZ$ is $3/2$ times as great as that on the radii, and the square on the line $KB$ is more than twice as great as that on $BZ$, it follows that the square on $KE$ is more than twice as great as that on $KZ$. The square on the line $EZ$ is $3/2$ times as great as that on $EK$. Therefore the square on $EZ$ is greater than those on $EK$ and
$KZ$ together. Therefore the angles at $K$ is obtuse, so that the segment in which it is inscribed is less than a semicircle.

In this manner Hippocrates squared every lunule, whether the outer boundary was a semicircle, or greater of less than a semicircle.

To square a lunule together with a circle, he proceeded as follows:

Let there be two circles with centers at $K$, and let the square of the diameters of the outer circle be 6 times as great as that of the inner circle; let a hexagon be inscribed in the inner circle and let the lines $KA$, $KB$ and $KT$ be then extended from the center until they meet the outer circle in $H$, $Θ$ adn $I$. And let a segment be described about the line $HI$ similar to the segment cut off by $HΘ$. Since the square on the line $HI$ is three times as great as that on the side $HΘ$ of the hexagon, which in turn is six times as great as that on $AB$, it follows that the segment constructed on $HI$ is equal to the sum of the segments cut off from the outer circle by the lines $HΘ$ and $ΘI$, together with (all the segments) cut off from the inner circle by the sides of the hexagon. Therefore the lunule $HΘI$ will be as much less than the triangle designated by the same letters as the segments cut off from the inner circle by the sides of the hexagon (are together). The lunule and the segments cut off by the hexagon are together equal to the triangle, And when the hexagon is added to each of these, it follows that the triangle and the hexagon are together equal to the sum of this lunule and the inner circle. By determining the area of these rectilinear figures, one can therefor also square the circle plus the lunule.
Chapter 3

Aristotle

3.1 On the Continuous and Zeno’s paradoxes

3.1.1 From the Metaphysics

Book XI

... Things which are in one place (in the strictest sense) are together in place, and things which are in different places are apart. Things whose extremes are together touch. That at which the changing thing, if it changes continuously according to its nature, naturally arrives before it arrives at the extreme into which it is changing, is between? That which is most distant in a straight line is contrary in place. That is successive which is after the beginning (the order being determined by position or form or in some other way) and has nothing of the same class between it and that which it succeeds, e.g. lines succeed a line, units a unit, or one house another house. (There is nothing to prevent a thing of some other class from being between.) For the successive succeeds something and is something later; 'one' does not succeed 'two', nor the first io6ga day of the month the second. That which, being successive, touches, is contiguous. Since all change is between bposites, and these are either contraries or contradictories, and there is no middle term for contradictories, clearly that which is between is between contraries. The continuous is a species of the contiguous or of that which touches; two things are called continuous when the limits of each, with which they touch and are kept together, become one and the same, so that plainly the continuous is found in the things out of which a unity naturally arises in virtue of their contact. And plainly the successive is the first of these concepts; for the successive does not necessarily touch, but that which 10 touches is successive. And if a thing is continuous, it touches, but if it touches, it is not necessarily continuous; and in things in which there is no touching, there is no organic unity. Therefore a point is not the same as a unit; for contact belongs to points, but not to units, which have only succession;
and there is something between two of the former, but not between two of the latter. 

3.1.2 From the Physics

Book VI Chapter 1

Now if the terms continuous, in contact, and in succession are understood as defined above things being continuous if their extremities are one, in contact if their extremities are together, and in succession if there is nothing of their own kind intermediate between them—nothing that is continuous can be composed of indivisibles: e.g. a line cannot be composed of points, the line being continuous and the point indivisible. For the extremities of two points can neither be one (since of an indivisible there can be no extremity as distinct from some other part) nor together (since that which has no parts can have no extremity, the extremity and the thing of which it is the extremity being distinct).

Moreover, if that which is continuous is composed of points, these points must be either continuous or in contact with one another: and the same reasoning applies in the case of all indivisibles. Now for the reason given above they cannot be continuous: and one thing can be in contact with another only if whole is in contact with whole or part with part or part with whole. But since indivisibles have no parts, they must be in contact with one another as whole with whole. And if they are in contact with one another as whole with whole, they will not be continuous: for that which is continuous has distinct parts: and these parts into which it is divisible are different in this way, i.e. spatially separate.

Nor, again, can a point be in succession to a point or a moment to a moment in such a way that length can be composed of points or time of moments: for things are in succession if there is nothing of their own kind intermediate between them, whereas that which is intermediate between points is always a line and that which is intermediate between moments is always a period of time.

Again, if length and time could thus be composed of indivisibles, they could be divided into indivisibles, since each is divisible into the parts of which it is composed. But, as we saw, no continuous thing is divisible into things without parts. Nor can there be anything of any other kind intermediate between the parts or between the moments: for if there could be any such thing it is clear that it must be either indivisible or divisible, and if it is divisible, it must be divisible either into indivisibles or into divisibles that are infinitely divisible, in which case it is continuous.

Moreover, it is plain that everything continuous is divisible into divisibles that are infinitely divisible: for if it were divisible into indivisibles, we should have an indivisible in contact with an indivisible, since the extremities of things that are continuous with one another are one and are in contact.

\(^1\)Metaphysics, 1068b-1069a. [RS], pages 21-22.
The same reasoning applies equally to magnitude, to time, and to motion: either all of these are composed of indivisibles and are divisible into indivisibles, or none. This may be made clear as follows. If a magnitude is composed of indivisibles, the motion over that magnitude must be composed of corresponding indivisible motions: e.g. if the magnitude $AB\Gamma$ is composed of the indivisibles $A$, $B$, $\Gamma$, each corresponding part of the motion $\Delta EZ$ of $O$ over $AB\Gamma$ is indivisible. Therefore, since where there is motion there must be something that is in motion, and where there is something in motion there must be motion, therefore the being-moved will also be composed of indivisibles. So $O$ traversed $A$ when its motion was $\Delta$, $B$ when its motion was $E$, and $\Gamma$ similarly when its motion was $Z$. Now a thing that is in motion from one place to another cannot at the moment when it was in motion both be in motion and at the same time have completed its motion at the place to which it was in motion: e.g. if a man is walking to Thebes, he cannot be walking to Thebes and at the same time have completed his walk to Thebes: and, as we saw, $O$ traverses a the partless section $A$ in virtue of the presence of the motion $\Delta$. Consequently, if $O$ actually passed through $A$ after being in process of passing through, the motion must be divisible: for at the time when $O$ was passing through, it neither was at rest nor had completed its passage but was in an intermediate state: while if it is passing through and has completed its passage at the same moment, then that which is walking will at the moment when it is walking have completed its walk and will be in the place to which it is walking: that is to say, it will have completed its motion at the place to which it is in motion. And if a thing is in motion over the whole $AB\Gamma$ and its motion is the three $\Delta$, $H$, and $Z$, and if it is not in motion at all over the partless section $A$ but has completed its motion over it, then the motion will consist of motions but of starts, and will take place by a thing having completed a motion without being in motion: for on this assumption it has completed its passage through $A$ without passing through it. So it will be possible for a thing to have completed a walk without ever walking: for on this assumption it has completed a walk over a particular distance without walking over that distance. Since, then, everything must be either at rest or in motion, and $O$ is therefore at rest in each of the sections $A$, $B$, and $\Gamma$, it follows that a thing can be continuously at rest and at the same time in motion: for, as we saw, $O$ is in motion over the whole $AB\Gamma$ and at rest in any part (and consequently in the whole) of it. Moreover, if the indivisibles composing $\Delta EZ$ are motions, it would be possible for a thing in spite of the presence in it of motion to be not in motion but at rest, while if they are not motions, it would be possible for motion to be composed of something other than motions.

And if length and motion are thus indivisible, it is neither more nor less necessary that time also be similarly indivisible, that is to say be composed of indivisible moments: for if the whole distance is divisible and an equal velocity will cause a thing to pass through less of it in less time, the time must also be divisible, and conversely, if the time in which a thing is carried over the section $A$ is divisible, this section $A$ must also be divisible.
Chapter 2

And since every magnitude is divisible into magnitudes for we have shown that it is impossible for anything continuous to be composed of indivisible parts, and every magnitude is continuous; it necessarily follows that the quicker of two things traverses a greater magnitude in an equal time, an equal magnitude in less time, and a greater magnitude in less time, in conformity with the definition sometimes given of the quicker. Suppose that \( A \) is quicker than \( B \). Now since of two things that which changes sooner is quicker, in the time \( ZH \), in which \( A \) has changed from \( \Gamma \) to \( \Delta \), \( B \) will not yet have arrived at \( \Delta \) but will be short of it: so that in an equal time the quicker will pass over a greater magnitude. More than this, it will pass over a greater magnitude in less time: for in the time in which \( A \) has arrived at \( \Delta \), \( B \) being the slower has arrived, let us say, at \( E \). Then since \( A \) has occupied the whole time \( ZH \) in arriving at \( \Delta \), \( A \) will have arrived at \( E \) in less time than this, say \( ZK \). Now the magnitude \( \Gamma \Delta \) that \( A \) has passed over is greater than the magnitude \( \Gamma E \), and the time \( ZK \) is less than the whole time \( ZH \): so that the quicker will pass over a greater magnitude in less time. And from this it is also clear that the quicker will pass over an equal magnitude in less time than the slower. For since it passes over the greater magnitude in less time than the slower, and (regarded by itself) passes over \( \Delta M \) the greater in more time than \( \Lambda \Xi \) the lesser, the time \( \Pi P \) in which it passes over \( \Lambda M \) will be more than the time \( \Pi \Sigma \), which it passes over \( \Lambda \Xi \): so that, the time \( \Pi P \) being less than the time \( \Pi X \) in which the slower passes over \( \Lambda \Xi \), the time \( \Pi \Sigma \) will also be less than the time \( \Pi \Xi \): for it is less than the time \( \Pi P \), and that which is less than something else that is less than a thing is also itself less than that thing. Hence it follows that the quicker will traverse an equal magnitude in less time than the slower. Again, since the motion of anything must always occupy either an equal time or less or more time in comparison with that of another thing, and since, whereas a thing is slower if its motion occupies more time and of equal velocity if its motion occupies an equal time, the quicker is neither of equal velocity nor slower, it follows that the motion of the quicker can occupy neither an equal time nor more time. It can only be, then, that it occupies less time, and thus we get the necessary consequence that the quicker will pass over an equal magnitude (as well as a greater) in less time than the slower.

And since every motion is in time and a motion may occupy any time, and the motion of everything that is in motion may be either quicker or slower, both quicker motion and slower motion may occupy any time: and this being so, it necessarily follows that time also is continuous. By continuous I mean that which is divisible into divisibles that are infinitely divisible: and if we take this as the definition of continuous, it follows necessarily that time is continuous. For since it has been shown that the quicker will pass over an equal magnitude in less time than the slower, suppose that \( A \) is quicker and \( B \) slower, and that the slower has traversed the magnitude \( \Gamma \Delta \) in the time \( ZH \). Now it is clear that the quicker will traverse the same magnitude in less time than this: let us say in the time \( ZO \). Again, since the quicker has passed over the whole \( \Gamma \Delta \) in the time \( ZO \), the slower will in the same time pass over \( \Gamma K \), say, which is less than \( \Gamma \Delta \).
And since \( B \), the slower, has passed over \( \Gamma K \) in the time \( ZO \), the quicker will pass over it in less time: so that the time \( ZO \) will again be divided. And if this is divided the magnitude \( \Gamma K \) will also be divided just as \( \Gamma \Delta \) was: and again, if the magnitude is divided, the time will also be divided. And we can carry on this process for ever, taking the slower after the quicker and the quicker after the slower alternately, and using what has been demonstrated at each stage as a new point of departure: for the quicker will divide the time and the slower will divide the length. If, then, this alternation always holds good, and at every turn involves a division, it is evident that all time must be continuous. And at the same time it is clear that all magnitude is also continuous; for the divisions of which time and magnitude respectively are susceptible are the same and equal.

Moreover, the current popular arguments make it plain that, if time is continuous, magnitude is continuous also, insasmuch as a thing passes over half a given magnitude in half the time taken to cover the whole: in fact without qualification it passes over a less magnitude in less time; for the divisions of time and of magnitude will be the same. And if either is infinite, so is the other, and the one is so in the same way as the other; i.e. if time is infinite in respect of its extremities, length is also infinite in respect of its extremities: if time is infinite in respect of divisibility, length is also infinite in respect of divisibility: and if time is infinite in both respects, magnitude is also infinite in both respects.

Hence Zeno's argument makes a false assumption in asserting that it is impossible for a thing to pass over or severally to come in contact with infinite things in a finite time. For there are two senses in which length and time and generally anything continuous are called infinite: they are called so either in respect of divisibility or in respect of their extremities. So while a thing in a finite time cannot come in contact with things quantitatively infinite, it can come in contact with things infinite in respect of divisibility: for in this sense the time itself is also infinite: and so we find that the time occupied by the passage over the infinite is not a finite but an infinite time, and the contact with the infinites is made by means of moments not finite but infinite in number.

The passage over the infinite, then, cannot occupy a finite time, and the passage over the finite cannot occupy an infinite time: if the time is infinite the magnitude must be infinite also, and if the magnitude is infinite, so also is the time. This may be shown as follows. Let \( AB \) be a finite magnitude, and let us suppose that it is traversed in infinite time \( \Gamma \), and let a finite period \( \Gamma \Delta \) of the time be taken. Now in this period the thing in motion will pass over a certain segment of the magnitude: let \( BE \) be the segment that it has thus passed over. (This will be either an exact measure of \( AB \) or less or greater than an exact measure: it makes no difference which it is.) Then, since a magnitude equal to \( BE \) will always be passed over in an equal time, and \( BE \) measures the whole magnitude, the whole time occupied in passing over \( AB \) will be finite: for it will be divisible into periods equal in number to the segments into which the magnitude is divisible. Moreover, if it is the case that infinite time is not occupied in passing over every magnitude, but it is possible to pass over some magnitude, say \( BE \), in a finite time, and if this \( BE \) measures the whole of which
it is a part, and if an equal magnitude is passed over in an equal time, then it follows that the time like the magnitude is finite. That infinite time will not be occupied in passing over $BE$ is evident if the time be taken as limited in one direction: for as the part will be passed over in less time than the whole, the time occupied in traversing this part must be finite, the limit in one direction being given. The same reasoning will also show the falsity of the assumption that infinite length can be traversed in a finite time. It is evident, then, from what has been said that neither a line nor a surface nor in fact anything continuous can be indivisible.

This conclusion follows not only from the present argument but from the consideration that the opposite assumption implies the divisibility of the indivisible. For since the distinction of quicker and slower may apply to motions occupying any period of time and in an equal time the quicker passes over a greater length, it may happen that it will pass over a length twice, or one and a half times, as great as that passed over by the slower: for their respective velocities may stand to one another in this proportion. Suppose, then, that the quicker has in the same time been carried over a length one and a half times as great as that traversed by the slower, and that the respective magnitudes are divided, that of the quicker, the magnitude $AB\Gamma\Delta$, into three indivisibles, and that of the slower into the two indivisibles $EZ, ZH$. Then the time may also be divided into three indivisibles, for an equal magnitude will be passed over in an equal time. Suppose then that it is thus divided into $K\Lambda, \Lambda M, MN$. Again, since in the same time the slower has been carried over $EZ, ZH$, the time may also be similarly divided into two. Thus the indivisible will be divisible, and that which has no parts will be passed over not in an indivisible but in a greater time. It is evident, therefore, that nothing continuous is without parts.

Chapter 9

Zenon reasoning, however, is fallacious, when he says that if everything when it occupies an equal space is at rest, and if that which is in locomotion is always occupying such a space at any moment, the flying arrow is therefore motionless. This is false, for time is not composed of indivisible moments any more than any other magnitude is composed of indivisibles.

Zenon arguments about motion, which cause so much disquietude to those who try to solve the problems that they present, are four in number. The first asserts the non-existence of motion on the ground that that which is in locomotion must arrive at the half-way stage before it arrives at the goal. This we have discussed above.

The second is the so-called Achilles, and it amounts to this, that in a race the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead. This argument is the same in principle as that which depends on bisection, though it differs from it in that the spaces with which we successively have to deal are not divided into halves. The result of the argument is that the
3.1. ON THE CONTINUOUS AND ZENO’S PARADOXES

slower is not overtaken: but it proceeds along the same lines as the bisection-argument (for in both a division of the space in a certain way leads to the result that the goal is not reached, though the Achilles goes further in that it affirms that even the quickest runner in legendary tradition must fail in his pursuit of the slowest), so that the solution must be the same. And the axiom that that which holds a lead is never overtaken is false: it is not overtaken, it is true, while it holds a lead: but it is overtaken nevertheless if it is granted that it traverses the finite distance prescribed. These then are two of his arguments.

The third is that already given above, to the effect that the flying arrow is at rest, which result follows from the assumption that time is composed of moments: if this assumption is not granted, the conclusion will not follow.

The fourth argument is that concerning the two rows of bodies, each row being composed of an equal number of bodies of equal size, passing each other on a race-course as they proceed with equal velocity in opposite directions, the one row originally occupying the space between the goal and the middle point of the course and the other that between the middle point and the starting-post. This, he thinks, involves the conclusion that half a given time is equal to double that time. The fallacy of the reasoning lies in the assumption that a body occupies an equal time in passing with equal velocity a body that is in motion and a body of equal size that is at rest; which is false. For instance (so runs the argument), let \( A, A \) be the stationary bodies of equal size, \( B, B \) the bodies, equal in number and in size to \( A, A \), originally occupying the half of the course from the starting-post to the middle of the \( A \)s, and \( \Gamma, \Gamma \) those originally occupying the other half from the goal to the middle of the \( A \)s, equal in number, size, and velocity to \( B, B \) Then three consequences follow:

First, as the \( B \)s and the \( \Gamma \)s pass one another, the first \( B \) reaches the last \( \Gamma \) at the same moment as the first \( \Gamma \) reaches the last \( B \). Secondly at this moment the first \( \Gamma \) has passed all the \( A \)s, whereas the first \( B \) has passed only half the \( A \)s, and has consequently occupied only half the time occupied by the first \( \Gamma \), since each of the two occupies an equal time in passing each \( A \). Thirdly, at the same moment all the \( B \)s have passed all the \( \Gamma \)s: for the first \( \Gamma \) and the first \( B \) will simultaneously reach the opposite ends of the course, since (so says Zeno) the time occupied by the first \( \Gamma \) in passing each of the \( B \)s is equal to that occupied by it in passing each of the \( A \)s, because an equal time is occupied by both the first \( B \) and the first \( \Gamma \) in passing all the \( A \)s. This is the argument, but it presupposed the aforesaid fallacious assumption.

Nor in reference to contradictory change shall we find anything unanswerable in the argument that if a thing is changing from not-white, say, to white, and is in neither condition, then it will be neither white nor not-white: for the fact that it is not wholly in either condition will not preclude us from calling it white or not-white. We call a thing white or not-white not necessarily because it is be one or the other, but cause most of its parts or the most essential parts of it are so: not being in a certain condition is different from not being wholly in that condition. So, too, in the case of being and not-being and all other conditions which stand in a contradictory relation: while the changing thing
must of necessity be in one of the two opposites, it is never wholly in either.

Again, in the case of circles and spheres and everything whose motion is confined within the space that it occupies, it is not true to say the motion can be nothing but rest, on the ground that such things in motion, themselves and their parts, will occupy the same position for a period of time, and that therefore they will be at once at rest and in motion. For in the first place the parts do not occupy the same position for any period of time: and in the second place the whole also is always changing to a different position: for if we take the orbit as described from a point $A$ on a circumference, it will not be the same as the orbit as described from $B$ or $\Gamma$ or any other point on the same circumference except in an accidental sense, the sense that is to say in which a musical man is the same as a man. Thus one orbit is always changing into another, and the thing will never be at rest. And it is the same with the sphere and everything else whose motion is confined within the space that it occupies.

### 3.2 On the Infinite

#### 3.2.1 From the Metaphysics

Book XI, Chapter X

The infinite is either that which is incapable of being traversed because it is not its nature to be traversed (this corresponds to the sense in which the voice is 'invisible'), or that which admits only of incomplete traverse or scarcely admits of traverse, or that which, though it naturally admits of traverse, is not traversed or limited: further, a thing may be infinite in respect of addition or of subtraction or of both. The infinite cannot be a separate, independent thing. For if it is neither a spatial magnitude nor a plurality, but infinity itself is its substance and not an accident, it will be indivisible; for the divisible is either magnitude or plurality. But if indivisible, it is not infinite, except as the voice is invisible; but people do not mean this, nor are we examining this sort of infinite, but the infinite as untraversable. Further, how can an infinite exist by itself, unless number and magnitude also exist by themselves, since infinity is an attribute of these? Further, if the infinite is an accident of something else, it cannot be qua infinite an element in things, as the invisible is not an element in speech, though the voice is invisible. And evidently the infinite cannot exist actually. For then any part of it that might be taken would be infinite; for 'to be infinite' and 'the infinite' are the same, if the infinite is substance and not predicated of a subject. Therefore it is either indivisible, or if it is secable, it is divisible into ever divisible parts; but the same thing cannot be many infinites, yet as a part of air is air, so a part of the infinite would be infinite, if the infinite is a substance and a principle. Therefore it must be inseparable and indivisible. But the actually infinite cannot be indivisible; for it must be a quantity. Therefore

\[\text{\[HG\]}\]
infinity belongs to a subject incidentally. But if so, as we have said, it cannot be it that is a principle, but rather that of which it is an accident the air or the even number.

This inquiry is universal; but that the infinite is not among sensible things, is evident from the following argument. If the definition of a body is ‘that which is bounded by planes’, there cannot be an infinite body either sensible or intelligible; nor a separate and infinite number, for number or that which has a number can be completely enumerated. The truth is evident from the following concrete argument. The infinite can neither be composite nor simple. For (1) it cannot be a composite body, since the elements are limited in multitude. For the contraries must be equal and no one of them must be infinite; for if one of the two bodies falls at all short of the other in potency, the finite will be destroyed by the infinite. And that each should be infinite is impossible. For body is that which has extension in all directions, and the infinite is the boundlessly extended, so that the infinite body will be infinite in every direction. Nor (2) can the infinite body be one and simple either, as some say, something which is apart from the elements, from which they generate these4 (for there is no such body apart from the elements; for everything can be resolved into that of which it consists, but no such product of analysis is observed except the simple bodies), nor fire nor any other of the elements. For apart from the question how any of them could be infinite, the All, even if it is finite, cannot either be or become one of them, as Heraclitus says all things sometime become fire. The same argument applies to the One, which the natural philosophers posit besides the elements. For everything changes from the contrary, e.g., from hot to cold.

Further, every sensible body is somewhere, and whole and part have the same proper place, e.g., the whole earth and part of the earth. Therefore if (1) the infinite body is homogeneous, it will be unmovable or it will be always moving. But the latter is impossible; for why should it rather move down than up or anywhere else? E.g., if there is a clod which is part of an infinite body, where will this move or rest? The proper place of the body which is homogeneous with it is infinite. Will the clod occupy the whole place, then? And how? (This is impossible.) What then is its rest or its movement? It will either rest everywhere, and then it cannot move; or it will move everywhere, and then it cannot be still. But (2) if the infinite body has unlike parts, the proper places of the parts are unlike also, and, firstly, the body of the All is not one except by contact, and, secondly, the parts will be either finite or infinite in variety of kind. Finite they cannot be; for then those of one kind will be infinite in quantity and those of another will not (if the All is infinite), e.g., fire or water would be infinite, but such an infinite part would be destruction to its contrary. But if the parts are infinite and simple, their places also are infinite and the elements will be infinite; and if this is impossible, and the places are finite, the All also must be limited.

In general, there cannot be an infinite body and also a proper place for all bodies, if every sensible body has either weight or lightness. For it must move either towards the middle or upwards, and the infinite either the whole or the
half cannot do either; for how will you divide it? Or how will part of the infinite be up and part down, or part extreme and part middle? Further, every sensible body is in a place, and there are six kinds of place, but these cannot exist in an infinite body. In general, if there cannot be an infinite place, there cannot be an infinite body; (and there cannot be an infinite place,) for that which is in a place is somewhere, and this means either up or down or in one of the other directions, and each of these is a limit.

The infinite is not the same in the sense that it is one thing whether exhibited in distance or in movement or in time, but the posterior among these is called infinite in virtue of its relation to the prior, i.e. a movement is called infinite in virtue of the distance covered by the spatial movement or alteration or growth, and a time is called infinite because of the movement which occupies it.

3.2.2 From the Physics

Book III, Chapter 1

[M]otion is supposed to belong to the class of things which are continuous; and the infinite presents itself first in the continuous – that is how it comes about that 'infinite' is often used in definitions of the continuous ('what is infinitely divisible is continuous')

Book III, Chapter 4

The science of nature is concerned with spatial magnitudes and motion and time, and each of these at least is necessarily infinite or finite, even if some things dealt with by the science are not, e.g. a quality or a point is not necessary perhaps that such things should be put under either head. Hence it is incumbent on the person who specializes in physics to discuss the infinite and to inquire whether there is such a thing or not, and, if there is, what it is.

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4[HG], page 31.
5[HG], page 34.
Chapter 4

Archimedes, c. 220 BC

4.1 Area of Circle

MEASUREMENT OF A CIRCLE

Proposition 1.

The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle.

Let $ABCD$ be the given circle, $K$ the triangle described.

Then, if the circle is not equal to $K$, it must be either greater or less.

I. If possible, let the circle be greater than $K$.

Inscribe a square $ABCD$, bisect the arcs $AB$, $BC$, $CD$, $DA$, then bisect (if necessary) the halves, and so on, until the sides of the inscribed polygon whose angular points are the points of division subtend segments whose sum is less than the excess of the area of the circle over $K$.

Thus the area of the polygon is greater than $K$.

Let $AE$ be any side of it, and $ON$ the perpendicular on $AE$ from the centre $O$.

Then $ON$ is less than the radius of the circle and therefore less than one of the sides about the right angle in $K$. Also the perimeter of the polygon is less than the circumference of the circle, i.e. less than the other side about the right angle in $K$.

Therefore the area of the polygon is less than $K$; which is inconsistent with the hypothesis.

Thus the area of the circle is not greater than $K$.

II. If possible, let the circle be less than $K$.

Circumscribe a square, and let two adjacent sides, touching the circle in $E$, $H$, meet in $T$. Bisect the arcs between adjacent points of contact and draw the tangents at the points of bisection. Let $A$ be the middle point of the arc $EH$, and $FAO$ the tangent at $A$.

Then the angle $TAG$ is a right angle.

Therefore $TG > GA > GH$.

It follows that the triangle $FTG$ is greater than half the area $TEAK$.

Similarly, if the arc $AH$ be bisected and the tangent at the point of bisection be drawn, it will cut off from the area $GAB$ more than one-half.

Thus, by continuing the process, we shall ultimately arrive at a circumscribed polygon such that the spaces intercepted between it and the circle are together less than the excess of $K$ over the area of the circle.

Thus the area of the polygon will be less than $K$.

Now, since the perpendicular from $O$ on any side of the polygon is equal to the radius of the circle, while the perimeter of the polygon is greater than the circumference of the circle, it follows that the area of the polygon is greater than the triangle $K$; which is impossible.

Therefore the area of the circle is not less than $K$. Since then the area of the circle is neither greater nor less than $K$, it is equal to it.

### 4.2 Postulate of Archimedes

Assumptions from “On the Sphere and Cylinder”

1. Of all lines which have the same extremities the straight line is the least.
2. Of other lines in a plane and having the same extremities, [any two] such are unequal whenever both are concave in the same direction and one of them is either wholly included between the other and the straight line which has the same extremities with it, or is partly included by, and is partly common with, the other; and that [line] which is included is the lesser [of the two].

3. Similarly, of surfaces which have the same extremities, if those extremities are in a plane, the plane is the least [in area]. 4. Of other surfaces with the same extremities, the extremities being in a plane, [any two] such are unequal whenever both are concave in the same direction and one surface is either wholly included between the other and the plane which has the same extremities with it, or is partly included by, and partly common with, the other; and that [surface] which is included is the lesser [of the two in area].

5. Further, of unequal lines, unequal surfaces, and unequal solids, the greater exceeds the less by such a magnitude as, when added to itself, can be made to exceed any assigned magnitude among those which are comparable with [it and with] one another.

These things being premised, if a polygon be inscribed in a circle, it is plain that the perimeter of the inscribed polygon is less than the circumference of the circle; for each of the sides of the polygon is less than that part of the circumference of the circle which is cut off by it.

4.3 From “Quadrature of the Parabola”

Archimedes to Dositheus greeting[3]

When I heard that Conon, who was my friend in his lifetime, was dead, but that you were acquainted with Conon and withal versed in geometry, while I grieved for the loss not only of a friend but of an admirable mathematician, I set myself the task of communicating to you, as I had intended to send to Conon, a certain geometrical theorem which had not been investigated before but has now been investigated by me, and which I first discovered by means of mechanics and then exhibited by means of geometry. Now some of the earlier geometers tried to prove it possible to find a rectilineal area equal to a given circle and a given segment of a circle; and after that they endeavoured to square the area bounded by the section of the whole cone and a straight line, assuming lemmas not easily conceded, so that it was recognised by most people that the problem was not solved. But I am not aware that any one of my predecessors has attempted to square the segment bounded by a straight line and a section of a rightangled cone [a parabola], of which problem I have now discovered the solution. For it is here shown that every segment bounded by a straight line and a section of a right-angled cone [a parabola] is four-thirds of the triangle which has the same base and equal height with the segment, and for the demonstration of this property the following lemma is assumed: that the excess by which the

greater of (two) unequal areas exceeds the less can, by being added to itself, be made to exceed any given finite area. The earlier geometers have also used this lemma; for it is by the use of this same lemma that they have shown that circles are to one another in the duplicate ratio of their diameters, and that spheres are to one another in the triplicate ratio of their diameters, and further that every pyramid is one third part of the prism which has the same base with the pyramid and equal height; also, that every cone is one third part of the cylinder having the same base as the cone and equal height they proved by assuming a certain lemma similar to that aforesaid. And, in the result, each of the aforesaid theorems has been accepted no less than those proved without the lemma. As therefore my work now published has satisfied the same test as the propositions referred to, I have written out the proof and send it to you, first as investigated by means of mechanics, and afterwards too as demonstrated by geometry. Prefixed are, also, the elementary propositions in conies which are of service in the proof. Farewell.

... Proposition 20.

If \(Qq\) be the base, and \(P\) the vertex, of a parabolic segment, then the triangle \(PQq\) is greater than half the segment \(PQq\).

For the chord \(Qq\) is parallel to the tangent at \(P\), and the triangle \(PQq\) is half the parallelogram formed by \(Qq\), the tangent at \(P\), and the diameters through \(Q\), \(q\).

Therefore the triangle \(PQq\) is greater than half the segment.

Cor. It follows that it is possible to inscribe in the segment a polygon such that the segments left over are together less than any assigned area.

... Proposition 23.

Given a series of areas \(A, B, C, D, \ldots, Z\), of which \(A\) is the greatest, and each is equal to four times the next in order, then

\[
A + B + C + \ldots + Z + \frac{1}{3}Z = \frac{4}{3}A.
\]

Take areas \(b, c, d, \ldots\) such that \(b = 1/3B\), \(c = 1/3C\), \(d = 1/3D\), and so on.

Then, since \(b = 1/3B\) and \(B = 1/4A\), \(B+b = 1/3A\). Similarly \(C+c = 1/3B\).

Therefore

\[
B + C + D + \ldots + Z + b + c + d + \ldots + z = 1/3(A + B + C + \ldots + Y).
\]

But

\[
b + c + d + \ldots + y = 1/3(B + C + D + \ldots + Y).
\]
4.3. FROM “QUADRATURE OF THE PARABOLA”

Therefore, by subtraction,

\[ B + C + D + \ldots + Z + z = \frac{1}{3}A \]

or

\[ A + B + C + D + \ldots + Z + \frac{1}{3}Z = \frac{4}{3}A \]

**Proposition 24.**

*Every segment bounded by a parabola and a chord Qq is equal to four-thirds of the triangle which has the same base as the segment and equal height.*

Suppose \( K = \frac{4}{3}\Delta PQq \), where \( P \) is the vertex of the segment; and we have then to prove that the area of the segment is equal to \( K \).

For, if the segment be not equal to \( K \), it must either be greater or less.

I. Suppose the area of the segment greater than \( K \).

If then we inscribe in the segments cut off by \( PQ \), \( Pq \) triangles which have the same base and equal height, i.e. triangles with the same vertices \( R, r \) as those of the segments, and if in the remaining segments we inscribe triangles in the same manner, and so on, we shall finally have segments remaining whose sum is less than the area by which the segment \( PQq \) exceeds \( K \).

Therefore the polygon so formed must be greater than the area \( K \); which is impossible, since [Prop. 23]

\[ A + B + C + \ldots + Z < \frac{4}{3}A, \]

where \( A = \Delta PQq \).

Thus the area of the segment cannot be greater than \( K \).

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4 Heath: The algebraical equivalent of this result is of course \( 1 + \frac{1}{4} + (1/4)^2 + \ldots + (1/4)^{n-1} = \frac{4}{3} - 1/3(1/4)^{n-1} = \frac{1-(1/4)^n}{1-1/4} \).
II. Suppose, if possible, that the area of the segment is less than $K$.

If then $\Delta PQq = A$, $B = \frac{1}{4} A$, $C = \frac{1}{4} B$, and so on, until we arrive at an area $X$ such that $X$ is less than the difference between $K$ and the segment, we have

$$A + B + C + \ldots + X + \frac{1}{3} X = \frac{4}{3} A = K.$$ [Prop. 23]

Now, since $K$ exceeds $A + B + C + \ldots + X$ by an area less than $X$, and the area of the segment by an area greater than $X$, it follows that

$$A + B + C + \ldots + Z > (\text{the segment});$$

which is impossible, by Prop. 22 above. Hence the segment is not less than $K$.

Thus, since the segment is neither greater nor less than $K$,

$$(\text{area of segment } PQq) = K = \frac{4}{3} \Delta PQq.$$

### 4.4 Area of Ellipse

From “On Conoids and Spheroids.”

**Proposition 4.**

The area of any ellipse is to that of the auxiliary circle as the minor axis to the major.

Let $AA'$ be the major and $BB'$ the minor axis of the ellipse, and let $BB'$ meet the auxiliary circle in $b, V$.

Suppose $O$ to be such a circle that

$$(\text{circle } AbA'b') : O = CA : CB.$$ 

Then shall $O$ be equal to the area of the ellipse.

For, if not, $O$ must be either greater or less than the ellipse.

I. If possible, let $O$ be greater than the ellipse.

We can then inscribe in the circle $O$ an equilateral polygon of $4n$ sides such that its area is greater than that of the ellipse, [cf. On the Sphere and Cylinder, I. 6.]

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5Heath, ibid. Pages 113-114.
Let this be done, and inscribe in the auxiliary circle of the ellipse the polygon $AefbgA'$... similar to that inscribed in $O$. Let the perpendiculars $eM, fN, ...$ on $AA'$ meet the ellipse in $E, F, ...$ respectively. Join $AE, EF, FB, ...$

Suppose that $P'$ denotes the area of the polygon inscribed in the auxiliary circle, and $P$ that of the polygon inscribed in the ellipse.

Then, since all the lines $eM, fN, ...$ are cut in the same proportions at $E, F, ...$, i.e.

$$eM : EM = fN : FN = ... = bC : BC,$$

the pairs of triangles, as $eAM, EAM$, and the pairs of trapeziums, as $eMNf, EMNF$, are all in the same ratio to one another as $bC$ to $BC$, or as $CA$ to $CB$.

Therefore, by addition,

$$P' : P = CA : CB.$$

Now

$$P' : (\text{polygon inscribed in } O) = (\text{circle } AbA'b') : O = CA : CB,$$

by hypothesis.

Therefore $P$ is equal to the polygon inscribed in $O$.

But this is impossible, because the latter polygon is by hypothesis greater than the ellipse, and a fortiori greater than $P$.

Hence $O$ is not greater than the ellipse.

II. If possible, let $O$ be less than the ellipse.

In this case we inscribe in the ellipse a polygon $P$ with $4n$ equal sides such that $P > O$.

Let the perpendiculars from the angular points on the axis $AA'$ be produced to meet the auxiliary circle, and let the corresponding polygon ($P'$) in the circle be formed.

Inscribe in $O$ a polygon similar to $P'$.

Then

$$P' : P = CA : CB = (\text{circle } AbA'b') : O,$$

by hypothesis, $= P' : (\text{polygon inscribed in } O)$.

Therefore the polygon inscribed in $O$ is equal to the polygon $P$; which is impossible, because $P > O$.

Hence $O$, being neither greater nor less than the ellipse, is equal to it; and the required result follows.
Chapter 5

Pre-Renaissance Pre-Cursors

5.1 Al-Haitham, Sums of Powers, c. 1000

Al-Haitham c. 1000. Sums of squares, cubes, etc. [ref 13, 13.1]

Could skip this in favor of Pascal. ?? Unless we use it as an excuse to learn sigma notation. 'Cause then could translate all future sums into reasonable notation. Maybe good idea.

Also, playing with it to find formulas for sums of powers would be a good algebra experience. Ok, I'm sold. Do it.

5.2 Uniform and Nonuniform Motion and The Merton College Mean Speed Theorem


[Part VI. Local Motion]
[Prologue]

There are three categories or generic ways in which motion, in the strict sense, can occur. For whatever is moved, is changed either in its place, or in its quantity, or in its quality. And since, in general, any successive motion whatever is fast or slow, and since no single method of determining velocity is applicable in the same sense to all three kinds of motion, it will be suitable to show how any change of this sort may be distinguished from another change of its own kind, with respect to speed or slowness. And because local motion is prior in nature to the other kinds, as the primary kind, we will carry out our intention

\footnote{[Gr] pp.237-241}
in this section, with respect to local motion, before treating of the other kinds.

[1. Measure of Uniform Velocity]

Although change of place is of diverse kinds, and is varied according to several essential as well as accidental differences, yet it will suffice for our purposes to distinguish uniform motion from nonuniform motion. Of local motions, then, that motion is called uniform in which an equal distance is continuously traversed with equal velocity in an equal part of time. Nonuniform motion can, on the other hand, be varied in an infinite number of ways, both with respect to the magnitude, and with respect to the time.

In uniform motion, then, the velocity of a magnitude as a whole is in all cases measured (metietur) by the linear path traversed by the point which is in most rapid motion, if there is such a point. And according as the position of this point is changed uniformly or nonuniformly, the complete motion of the whole body is said to be uniform or difform (nonuniform). Thus, given a magnitude whose most rapidly moving point is moved uniformly, then, however much the remaining points may be moving nonuniformly, that magnitude as a whole is said to be in uniform movement.

[2. Measure of Nonuniform Velocity]

In nonuniform motion, however, the velocity at any given instant will be measured (attendetur) by the path which would be described by the most rapidly moving point if, in a period of time, it were moved uniformly at the same degree of velocity (uniformiter illo gradu velocitatis) with which it is moved in that given instant, whatever [instant] be assigned. For suppose that the point A will be continuously accelerated throughout an hour. It is not then necessary that, in any instant of that hour as a whole, its velocity be measured by the line which that point describes in that hour. For it is not required, in order that any two points or any other two moving things be moved at equal velocity, that they should traverse equal spaces in an equal time; but it is possible that they traverse unequal spaces, in whatever proportion you may please. For suppose that point A is moved continuously and uniformly at C degrees of velocity, for an hour, and that it traverses a distance of a foot. And suppose that point B commences to move, from rest and in the first half of that hour accelerates its velocity to C degrees, while in the second half hour it decelerates from this velocity to rest. It is then found that at the middle instant of the whole hour point B will be moving at C degrees of velocity, and will fully equal the velocity of the point A. And yet, at the middle instant of that hour, B will not have traversed as long a line as A, other things being equal. In similar manner, the point B, traversing a finite line as small as you please, can be accelerated in its motion beyond any limit; for, in the first proportional part of that time, it may have a certain velocity, and in the second proportional part, twice that velocity, and in the third proportional part, four times that velocity, and so on without limit.

From this it clearly follows, that such a nonuniform or instantaneous velocity (velocitas instantanea) is not measured by the distance traversed, but by the
distance which would be traversed by such a point, if it were moved uniformly over such or such a period of time at that degree of velocity with which it is moved in that assigned instant.

3. Measure of Uniform Acceleration

With regard to the acceleration (\textit{intensio}) and deceleration (\textit{remisso}) of local motion, however, it is to be noted that there are two ways in which a motion may be accelerated or decelerated: namely, uniformly, or nonuniformly. For any motion whatever is \textit{uniformly accelerated} (\textit{uniformiter intenditur}) if, in each of any equal parts of the time whatsoever, it acquires an equal increment (\textit{latitudo}) of velocity. But a motion is \textit{nonuniformly accelerated or decelerated}, when it acquires or loses a greater increment of velocity in one part of the time than in another equal part.

In view of this, it is sufficiently apparent that when the latitude of motion or velocity is infinite, it is impossible for any body to acquire that latitude uniformly, in any finite time. And since any degree of velocity whatsoever differs by a finite amount from zero velocity, or from the privative limit of the intensive scale, which is rest therefore any mobile body may be uniformly accelerated from rest to any assigned degree of velocity; and likewise, it may be decelerated uniformly from any assigned velocity, to rest. And, in general, both kinds of change may take place uniformly, from any degree of velocity to any other degree.

In this connection, it should be noted that just as there is no degree of velocity by which, with continuously uniform motion, a greater distance is traversed in one part of the time than in another equal part of the time, so there is no latitude (i.e., increment, \textit{latitudo}) of velocity between zero degree [of velocity] and some finite degree, through which a greater distance is traversed by uniformly accelerated motion in some given time, than would be traversed in an equal time by a uniformly decelerated motion of that latitude. For whether it commences from zero degree or from some [finite] degree [of velocity], every latitude, as long as it is terminated at some finite degree, and as long as it is acquired or lost uniformly, will correspond to its mean degree [of velocity]. Thus the moving body, acquiring or losing this latitude uniformly during some assigned period of time, will traverse a distance exactly equal to what it would traverse in an equal period of time if it were moved uniformly at its mean degree [of velocity].

2. For of every such latitude commencing from rest and terminating at some [finite] degree [of velocity], the mean degree is one-half the terminal degree [of velocity] of that same latitude.

3. From this it follows that the mean degree of any latitude bounded by two degrees (taken either inclusively or exclusively) is more than half the more intense degree bounding that latitude.

4. From the foregoing it follows that when any mobile body is uniformly accelerated from rest to some given degree [of velocity], it will in that time traverse one-half the distance that it would traverse if, in that same time, it were moved uniformly at the degree [of velocity] terminating that latitude. For
that motion, as a whole, will correspond to the mean degree of that latitude, which is precisely one-half that degree which is its terminal velocity.

5. It also follows in the same way that when any moving body is uniformly accelerated from some degree of velocity (taken exclusively) to another degree inclusively or exclusively, it will traverse more than one-half the distance which it would traverse with a uniform motion, in an equal time, at the degree of velocity at which it arrives in the accelerated motion. For that whole motion will correspond to its mean degree of velocity, which is greater than one-half of the degree of velocity terminating the latitude to be acquired; for although a nonuniform motion of this kind will likewise correspond to its mean degree of velocity contained in this latitude being acquired, and, likewise, it will be as slow.

6. To prove, however, that in the case of acceleration from rest to a finite degree of velocity, the mean degree of velocity is exactly one-half the terminal degree of velocity, it should be known that if any three terms are in continuous proportion, the ratio of the first to the second, or of the second to the third, will be the same as the ratio of the difference between the first and the middle, to the difference between the middle and the third; as when the terms are 4, 9, 6, 4. For as 4 is to 2, or as 2 is to 1, so is the proportion of the difference between 4 and 2 to the difference between 2 and 1, because the difference between 4 and 2 is 2, while that between 2 and 1 is 1; and so with the other cases.

Let there be assigned, then, some term under which there is an infinite series of other terms which are in continuous proportion according to the ratio 2 to 1. Let each term be considered in relation to the one immediately following it. Then, whatever is the difference between the first term assigned and the second, such precisely will be the sum of all the differences between the succeeding terms. For whatever is the amount of the first proportional part of any continuum or of any finite quantity, such precisely is the amount of the sum of all the remaining proportional parts of it.

Since, therefore, every latitude is a certain quantity, and since, in general, in every quantity the mean is equidistant from the extremes, so the mean degree of any finite latitude whatsoever is equidistant from the two extremes, whether these two extremes be both of them positive degrees, or one of them be a certain degree and the other a privation of it or zero degree.

But, as has already been shown, given some degree under which there is an infinite series of other degrees in continuous proportion, and letting each term be considered in relation to the one next to it, then the difference or latitude between the first and the second degree, the one, namely, that is half the first will be equal to the latitude composed of all the differences or latitudes between all the remaining degrees, namely those which come after the first two. Hence, exactly equally and by an equal latitude that second degree, which is related to the first as a half to its double, will differ from that double as that same degree differs from zero degree or from the opposite extreme of the given magnitude.
And so it is proved universally for every latitude commencing from zero degree and terminating at some finite degree, and containing some degree and half that degree and one-quarter of that degree, and so on to infinity, that its mean degree is exactly one-half its terminal degree. Hence this is not only true of the latitude of velocity of motion commencing from zero degree [of velocity], but it could be proved and argued in just the same way in the case of latitudes of heat, cold, light, and other such qualities.

7. With respect, however, to the distance traversed in a uniformly accelerated motion commencing from zero degree [of velocity] and terminating at some finite degree [of velocity], it has already been said that the motion as a whole, or its whole acquisition, will correspond to its mean degree [of velocity]. The same thing holds true if the latitude of motion is uniformly acquired from some degree [of velocity] in an exclusive sense, and is terminated at some finite degree [of velocity].

From the foregoing it can be sufficiently determined for this kind of uniform acceleration or deceleration how great a distance will be traversed, other things being equal, in the first half of the time and how much in the second half. For when the acceleration of a motion takes place uniformly from zero degree [of velocity] to some degree [of velocity], the distance it will traverse in the first half of the time will be exactly one-third of that which it will traverse in the second half of the time.

And if, contrariwise, from that same degree [of velocity] or from any other degree whatsoever, there is uniform deceleration to zero degree [of velocity], exactly three times the distance will be traversed in the first half of the time, as will be traversed in the second half. For every motion as a whole, completed in a whole period of time, corresponds to its mean degree [of velocity] namely, to the degree it will have at the middle instant of the time. And the second half of the motion in question will correspond to the mean degree of the second half of that same motion, which is one-fourth of the degree [of velocity] terminating that latitude. Consequently, since this second half will last only through half the time, exactly one-fourth of the distance will be traversed in that second half as will be traversed in the whole motion. Therefore, of the whole distance being traversed by the whole motion, three-quarters will be traversed in the first half of the whole motion, and the last quarter will be traversed in its second half. It follows, consequently, that in this type of uniform intension and remission of a motion from some degree [of velocity] to zero degree, or from zero degree to some degree, exactly three times as much distance is traversed in the more intense half of the latitude as in the less intense half.

8. But any motion can be uniformly accelerated or decelerated from some degree [of velocity] to another degree in an endless number of ways, because it may be from some degree to a degree half of that, or to a degree one-fourth of it, or one-fifth, or to a degree two-thirds of that degree, or three quarters of it, and so on. Consequently there can be no universal numerical value by which one will be able to determine, for all cases, how much more distance would be traversed in the first half of this sort of acceleration or deceleration than in
the second half, because, according to the diversity of the extreme degrees [of velocity], there will be diverse proportions of distance traversed in the first half of the time to distance traversed in the second half.

But if the extreme degrees [of velocity] are determined, so that it is known, for instance, that so much distance would be traversed in such or such a time by a uniform motion at the more intense limiting degree [of velocity], and if this is likewise known with respect to the less intense limiting degree [of velocity], then it will be known by calculation how much would be traversed in the first half and also how much in the second. For, if the extreme degrees [of velocity] are known in this way, the mean degree [of velocity] of these can be obtained, and also the mean degree between that mean degree and the more intense degree terminating the latitude. But a calculation of this kind offers more difficulty than advantage.

And it is sufficient, therefore, for every case of this kind, to state as a general law, that more distance will be traversed by the more intense half of such a latitude than by the less intense half as much more, namely, as would be [the excess of distance] traversed by the mean degree [of velocity] of this more intense half, if it moved in a time equal to that in which this half is acquired or lost uniformly, over that [distance which] would be traversed by the mean degree [of velocity] of the less intense half, in the same time.

9. But as concerns nonuniform acceleration or deceleration, whether from some degree [of velocity] to zero degree or vice versa, or from one degree to some other degree, there can be no rule determining the distance traversed in such or such time, or determining the intrinsic degree to which such a latitude of motion, acquired or lost nonuniformly, will correspond. For just as such a nonuniform acceleration or deceleration could vary in an infinite number of ways, so also that motion as a whole could correspond to an infinite number of intrinsic degrees [of velocity] of its latitude indeed, to any intrinsic degree whatsoever, of the latitude thus acquired or lost.

In general, therefore, the degree [of velocity] terminating such a latitude at its more intense limit is the most remiss degree [of velocity], beyond the other limit (i.e., the most remiss extreme) of the latitude, to which such a nonuniformly nonuniform motion as a whole cannot correspond; and the degree [of velocity] terminating that latitude at its more remiss limit is the most intense degree [of velocity] beneath the upper limit of the same latitude, to which such a nonuniformly nonuniform motion cannot correspond. Consequently, it is not possible for such a motion as a whole to correspond to such a remiss degree (as that of the lower limit); nor to such an intense degree (as the upper limit).
5.3 Nicole Oresme

5.3.1 Questions on the Geometry of Euclid, c. 1350

Question 1

Concerning the book of Euclid [that is, the Elements], we inquire first about a certain statement by Campanus asserting that a magnitude decreases into infinity. First we inquire whether a magnitude decreases into infinity according to proportional parts.

For this question, these things must be noted: firstly, what has [already] been said, namely that proportional parts are parts of the same proportion; secondly, in how many ways can such parts be imagined; thirdly, how something can be divided into such parts; fourthly, assumptions and proportions.

As to the first, it must be noted that proportional parts are said to be in continued proportion and such a proportion is a similitude of ratios, as it is called in the comment [by Campanus] on the ninth definition of the fifth [book of Euclid], where it is said that such a [relation or similitude] is had between at least two ratios; and for this reason Euclid said that the least number of terms in which it [that is, similitude of ratios] is found is three, but he does not give a maximum number because it is a process [that continues] into infinity. It follows from this that it is not proper to speak of [a single] proportional part, nor of two proportional parts, but it is necessary that there be at least three and there can be an infinite number. Proportional parts are said to be continually proportional because the first [proportional part] is related to the second as the second to the third, and so on if more are taken.

As to the second point, I reply that a division can be made into such proportional parts in as many ways as there are continuous proportionals; and there are just as many continuous proportionals as there are ratios, namely infinitely many. For example, it can happen that the first [proportional part] is double the second, and the second double the third, and so on just as is commonly said about the division of a continuum; and it could happen that the first [proportional part] is triple the second and the second triple the third, and so on.

As to the third point, it is held that a line and any continuum can be divided into such [proportional] parts. As line [can be so divided] in two ways because there are two extremities of it and such parts can begin from either one. A surface [can be divided into proportional parts] in infinite ways, and similarly for a body.

As to the fourth point, a first supposition is assumed such that if any ratio were increased to infinity with the greater term unchanged, the smaller term

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2 [Gr], pages 131–135. Footnotes on Oreme based on SJC and those in Grant.
3 Campanus of Novara: “magnitude decreases into infinity, but in numbers this is not so.”
4 A sequence of proportional parts has the form $a, ar, ar^2, ar^3, \ldots, ar^n, \ldots$ and is now called a geometric sequence’. For example, $2, 4, 8, \ldots$ or $1, 1/3, 1/9, \ldots, (1/3)^n, \ldots$
5 That is, $(1/2)^n$, where $n = 1, 2, 3, 4, \ldots$
6 That is, $(1/3)^n$, where $n = 1, 2, 3, 4, \ldots$
A second supposition is that if to any ratio there is added so much, and again so much, and so on into infinity, this ratio will be increased into infinity; and this is common to all qualities.

A third supposition is this: that to any quantity an addition can be made by proportional parts; and by the same [supposition] a diminution can be made by proportional parts.

The first proposition is that if an aliquot part should be taken from some quantity, and from the first remainder such a part is taken, and from the second remainder such a part is taken, and so on into infinity, such a quantity would be consumed exactly – no more, no less – by such a mode of subtraction. This is proved because the whole that was originally assumed, and the first remainder, and the second remainder, and the third, and so on, are continually proportional, as could be proved when arguing with an altered ratio. Therefore, there is a certain ratio, and then so much [of it], and so on endlessly; consequently, such a ratio of the whole to the remainder increases to infinity, because, by the second supposition, it is composed of these [ratios]. And the other term, say the whole, is imagined as unchanged, so that, by the first supposition, the remainder is diminished into infinity, and consequently, the whole quantity is consumed exactly.

This corollary follows: If from any foot [length] there should be taken away a half foot, and then half of the remainder of this quantity, and then half of the next remainder, and so on into infinity, the foot length will be removed exactly by the [procedure].

A second corollary is that if one–thousandth part of a foot were taken away [or removed], then [if] one–thousandth part of the remainder of this foot [were removed], and so on into infinity, exactly one foot would be subtracted from this [original foot].

But this is doubted. Since exactly half of one foot and then half of the remainder of that foot, and so on into infinity, make one foot, let this whole [foot] be $a$; similarly, by the second corollary, one–thousandth part of this one foot, and one–thousandth of another [that is, the next] remainder, and so on into infinity, make one foot, let this be $b$. It is then obvious that $a$ and $b$ are equal. But it can be proved that they are not equal because the first part of $a$ is is greater than the first part of $b$; and the second part of $a$ [is greater] than the second [part] of $b$, and so on to infinity. Therefore, the whole $a$ is greater than then whole $b$. And this is confirmed, [for] if Socrates were moved over $a$ for one hour and Plato over $b$ [for one hour], and [if] they divide the hour by

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7 If $\frac{A}{B} \to \infty$, then either $A \to \infty$ or $B \to 0$. So if $A$ is held constant, then $B$ must to go to 0.

8 Compounding greater–to-lesser ratios

9 I.e., we have $A, qA, q^2A, q^3A, \ldots$ or $A, B, C, D, \ldots$ where $\frac{B}{A} = \frac{C}{B} = \frac{D}{C} = \ldots = q$. 

### Chapter 5. Pre-Renaissance Pre-Cursors

would be diminished into infinity. This is obvious, since a ratio between two [terms] can be increased into infinity in two ways: either by the augmentation to infinity of the greater term or by the diminution to infinity of the lesser term.

A second supposition is that if to any ratio there is added so much, and again so much, and so on into infinity, this ratio will be increased into infinity; and this is common to all qualities.
proportional parts and traverse \( a \) and \( b \), respectively, then Socrates is moved quicker than Plato in the first proportional part; and similarly in the second [proportional part], and so beyond. Socrates will therefore traverse a greater distance than Plato, so that \( a \) is a greater distance than \( b \). In response to this, I deny the antecedent, namely that the first part of \( a \) is greater [than the first part of \( b \), and the second part of \( a \) is greater than the second part of \( b \)], and so on. The reason [for denying the antecedent] is that although the first part of \( a \) is greater than the first part of \( b \), and the second part of \( a \) is greater than the second part of \( b \), this is not so into infinity, since one part will eventually be reached that will not be greater than the part of \( b \) to be compared, but smaller. In this way the response to the question is clear, because in the imagination any continuum can have infinite proportional parts; similarly, the first [part] can really be separated from the others in thought and then the second, and so on into infinity.

As for the argument to the contrary, I deny the consequent, and for proof I say that although any proportional part belongs to the same quantity as any other [proportional part], it is nevertheless not of the same quantity with that [part] with which it is of the same proportion. They are not equals, since it follows [that if] they are equals, therefore they are mutually equal to themselves.

**Question 2**

Next we inquire wheather an addition to any magnitude could be made into infinity by proportional parts. ...

For this question, it must be noted in the first place that there is a ratio of equality and it is between equals; another [kind of ratio] is a ratio of greater inequality; which is of greater to smaller, as 4 to 2; and another [kind] is [a ratio] of lesser inequality; which is of smaller to greater, as 2 to 4. And these names differ with respect to relative superposition and subposition, as is obvious in what has been said before; and by there are three ways in which an addition can be made to any quantity.

Secondly, it must be noted that if an addition were made to infinity by proportional parts in a ratio of equality or of greater inequality, the whole would become infinite; if, however, this addition should be made [by proportional parts] in a ratio of lesser inequality, the whole would never become infinite, even if the addition continued into infinity. As will be declared afterward, the reason is because the whole will bear a certain finite ratio to the first [magnitude] assumed [or taken] to which the addition is made.

Finally, it must be noted that every term smaller than another which bears to it [that is, to the greater] a fixed ratio is called a fraction or fractions, or part or parts; and this is obvious in the principles of the seventh book of Euclid, and it is denominated by two numbers, one of which is the numerator and the other the denominator, as is clear in the same place [that is, in the principles of Euclid’s seventh book]. For example, on is smaller than two and is called one–half of two and one–third of three, and so on; and two is called two–thirds of three and two–fifthsh of five, and they ought to be written in this way; and
the two is called the numerator, the five the denominator.

The first proposition is that if a one–foot quantity should be assumed and an addition were made to it into infinity according to a subdouble [that is, one–half] proportion so that one–half of one foot is added to it, then one–fourth [of one foot], then one–eight, and so on into infinity by squaring the halves, the whole will be exactly double the first [magnitude] assumed. This is obvious, because if these [very] parts were taken from any quantity, exactly double the first quantity would be taken, as is clear from the first question preceeding. By a parity of reasoning, then, if the parts were added [the whole would be exactly double the first].

The second proposition is this: If any quantity were assumed, say one foot, then let one–third as much be added [to it], and then [let] one–third of what was added [be added to the sum], and so on into infinity, the whole will be exactly 11/2 feet, namely in a sesquialterate ratio to the first quantity assumed. Furthermore, this rule should be known: We must see how much the second part falls short of the first part, and [how much] the third [falls shorts] of the second, and so on with the others, and to denominate this [difference] by its denomination, and then the ratio of the whole aggregate to the quntity [first] assumed will be just as a denominator to a numerator. For example, in what was [just] proposed, the second part, which is one–third of the first, falls short by two–thirds of the [quantity of the] first, so that the ratio of the whole to the first part, or what was assumed, is as 3 to 2 and this is a sesquialterate [ratio].

The third proposition is this: It is possible that an addition could be made, though not proportionally, to any quantity by ratios of lesser inequality, and yet the whole would become infinite; but if it were done proportionally, it would be finite, as was said. For example, let a one–foot quntity be assumed to which one–half of a foot is added during the first proportional part of an hour, then one–third of a foot in another [or next proportional part of an hour], then one–fourth [of a foot], then one–fifth, and so on into infinity following the series of [natural] numbers, I say that the whole would become infinity, which is proved as follows: There exist infinite parts of which any one will be greater than one–half foot and [therefore] the whole will be infinite. The antecedent is obviou, since 1/4 and 1/3 are greater than 1/2; similarly [ths cume of the parts] from 1/5 to 1/8 [is greater than 1/2] and [also the sum of the parts] from 1/9 to 1/16, and so on into infinity ...

5.4 Treatise on the Configurations of Qualities and Motions, 1356

III.vii On the measure of difform qualities and velocities

Every quality, if it is uniformly difform, is of the same quantity as would be the quality of the same or equal subject that is uniform according to the

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10 [Cl], pages ???
degree of the middle point of the same subject. I understand this to hold if the quality is linear. If it is a surface quality, [then its quantity is equal to that of a quality of the same subject which is uniform] according to the degree of the middle line; if corporeal, according to the degree of the middle surface, always understanding [these concepts] in a conformable way. This will be demonstrated first for a linear quality. Hence let there be a quality imaginable by $\Delta ABC$, the quality being uniformly difform and terminated at no degree in point $B$. [see Fig. 1(a)]. And let $D$ be the middle point of the subject line. The degree of this point, or its intensity, is imagined by line $DE$. Therefore, the quality which would be uniform throughout the whole subject at degree $DE$ is imaginable by rectangle $AFGB$, as is evident by the tenth chapter of the first part. Therefore, it is evident by the 26th [proposition] of [Book] I [of the Elements] of Euclid that the two small triangles $EFC$ and $EGB$ are equal. Therefore, the larger $\Delta BAC$, which designates the uniformly difform quality, and the rectangle $AFGB$, which designates the quality uniform in the degree of the middle point, are equal. Therefore the qualities imaginable by a triangle and a rectangle of this kind are equal. And this is what has been proposed. In the same way it can be argued for a quality uniformly difform terminated in both extremes at a certain degree, as would be the quality imaginable by quadrangle $ABCD$ [see Fig. 21(b)]. For let line $DE$ be drawn parallel to the subject base and $\Delta CED$ would be formed. Then let line $FG$ be drawn through the degree of the middle point which is equal and parallel to the subject base. Also, let line $GD$ be drawn. Then, as before, it will be proved that $\Delta CED = \square EFGD$. Therefore, with the common rectangle $AEDB$ added to both of them, the two total areas are equal, namely quadrangle $ACDB$, which designates the uniformly difform quality, and the rectangle $AFGB$, which would designate the quality uniform at the degree of the middle point of the subject $AB$. Therefore, by chapter ten of the first part, the qualities designatable by quadrangles of this kind are equal. It can be argued in the same way regarding a surface quality and also regarding a corporeal quality. Now one should speak of velocity in completely the same fashion as linear quality, so long as the middle instant of the time measuring a velocity of this kind is taken in place of the middle point [of the subject]. And so it is clear to which uniform quality or velocity a quality or velocity uniformly difform is equated. Moreover, the ratio of uniformly difform qualities and velocities is as the ratio of the simply uniform qualities or velocities to which they are equated. And we have spoken of the measure and ratio of these uniform [qualities and velocities] in the preceding chapter. Further, if a quality or velocity is difformly difform, and if it is composed of uniform or uniformly difform parts, it can be measured by its parts, whose measure has been discussed before. Now, if the quality is difform in some other way, e.g. with the difformity designated by a curve, then it is necessary to have recourse to the mutual mensuration of the curved figures, or to [the mensuration of] these [curved figures] with rectilinear figures; and this is another kind of speculation. Therefore what has been stated is sufficient

III.viii On the measure and intension to infinity of certain difformities
A finite surface can be made as long as we wish, or as high, by varying the extension without increasing the size. For such a surface has both length and breadth and it is possible for it to be increased in one dimension as much as we like without the whole surface being absolutely increased so long as the other dimension is diminished proportionally, and this is also true of a body. For example, in the case of a surface, let there be a surface of one square foot in area whose base line is $AB$; and let there be another surface, similar and equal to it, whose base line is $CD$. Let the latter surface be imagined to be divided on line $CD$ to infinity into parts continually proportional according to the ratio 2 to 1, with its base divided in the same way. Let $E$ be the first part, $F$ the second, $G$ the third, and so on for the other parts. Therefore, let the first of these parts, namely $E$, which is one half the whole surface, be taken and placed on top of the first surface towards the extremity $B$. Then upon this whole let the second part, namely $F$, be placed, and again upon the whole let the third part, namely $G$, be placed, and so on for the others to infinity. When this has been done, let base line $AB$ be imagined as being divided into parts continually proportional according to the ratio 2 to 1, proceeding towards $B$. And it will be immediately evident that on the first proportional part of line $AB$ there stands a surface one foot high, on the second a surface two feet high, on the third one three feet high, on the fourth four feet high, and so on to infinity, and yet the whole surface is only the two square feet previously given, without augmentation. And consequently the whole surface standing on line $AB$ is precisely four times its part standing on the first proportional part of the same line $AB$. Therefore, that quality or velocity which would be proportional in intensity to this figure in altitude would be precisely four times the part of it which would be in the first part of the time or the subject so divided. For example, let the first part (towards extreme $A$) of the proportional parts divided along $AB$ according to the ratio 2 to 1 be a certain amount white or hot, the second twice as white [intensively], the third three times as white, the fourth four times, and so on to infinity on both sides according to the [natural] series of [whole] numbers. Then from the prior statements it is apparent that the total whiteness of line $AB$ is precisely four times the whiteness of the first part; and it would be the same for a surface, or for a corporeal, whiteness, if it were increased in intensity in similar fashion.

In the same way, if some mobile were moved with a certain velocity in the first proportional part of some period of time, divided in such a way, and in the second part it were moved twice as rapidly, and in the third three times as fast, in the fourth four times, and increasing in this way successively to infinity, the total velocity would be precisely four times the velocity of the first
part, so that the mobile in the whole hour would traverse precisely four times what it traversed in the first half of the hour; e.g., if in the first half or first proportional part it would traverse one foot, in the whole remaining period it would traverse three feet and in the total time it would traverse four feet. Moreover I have demonstrated this elsewhere with a more subtle and difficult demonstration. But this present treatment confirms more to [the scope of] this treatise, and suffices. Therefore I shall pass over the other [demonstration].

III.xi On the measure and extension to finity of a finite quality or velocity.  

Let us assume again the figure of chapter eight of this part, the figure whose baseline was \(AB\), and let us turn the image about or invert the figure so that its base of length is line \(BC\) infinitely extended beyond \(C\), and line \(BA\) is the first and greatest altitude of this figure. Then, as was shown in chapter eight, the whole surface or figure is precisely quadruple to half of the surface which lies on the first part of the line, which part is \(BC\). Therefore, this whole surface is double the surface which lies on \(BC\).

This is clear in another way. For the whole surface has [an] infinite [number] of parts, of which the second is as long as the first, the third as the second or first, and so on for the others, and the second is one half the first in altitude, and the third the second, and so on until the end. Therefore the first is double the second, the second is double the third, and so on for the others. Therefore, the whole is precisely double the first part. And the conclusion of the eighth chapter can be proved conversely by this argument.

From this it is clear that the quality of a subject infinitely extended which would be imaginable by this figure would be precisely double the quality of the first uniform art of that quality. Similarly, if some mobile would be moved during one day with a certain velocity, and during the second day twice as slowly, and during the third day twice as slowly as during the second day, and so on to infinity, never in eternity would it traverse twice that which it would traverse during the first day. But it would sometimes traverse a quantity of space equal to any given space less than double that traversed on the first day.

III.xii On the qualified infinite extension and measure of a finite and uniform quality

A corporeal quality has three dimension of subject. The figuration of such a subject or quality can be varied in many ways without its augmentation.

For example, let \(A\) be a body of a foot quantity divided by the designation of parts continually proportional according to a ratio of 2 to 1. Let \(B\) be the first...
part, C the second, D the third, and so on. And so let the first part be taken and a circle [i.e., a cylinder] be made of it. Then let the second be taken and added to the first circularly. But, however, let the width of the second be so increase that the total circle is twice as extended as before; and this second part is proportionally less deep so that it [i.e., the increase in extension] can take place without any augmentation and without rarefaction of that second part, i.e., so that it can take place by tranfiguration alone. Then the third proportional part of the body, i.e., D, is added and the total circle is made three times as wide [as it was a first], and the fourth part is added to make circle four times as wide; and afterwards a fifth [is added] in the same way, and similarly for the other parts, and this is always done without augmentation and without rarefaction, [i.e., it is done] by tranfiguration alone. With this done, it is immediately evident that body A will be infinitely long and infinitely wide and yet it will not be augmented but will be equal precisely to a foot.

Now in regard to the subject at hand, it is possible in the same way to imagine that some finite quality, e.g., gravity of one pound, is infinitely long and infinitely wide, and in addition that it is everywhere uniform or uniformly intense, since the subject which is uniformly heavy and one pound in weight can be transfigured and extended in the aforesaid way without alteration or the intensive decrease or increase of [its] gravity. It is possible, therefore, for such a quality extended in this way to be uniform ...

Also in the aforesaid body there would be an infinite surface whose uniform quality would be designatable by a body absolutely infinite – although not infinite in all dimensions; and yet the total corporeal quality would [itself] be absolutely finite. Thence the argument could be advanced that aoint is not somthing, nor is a line, nor is a surface, as was argued in the fourth chapter of this part. Furthermore, it could also be argued from this [case] that it is necessary for an agent to act according to its depth adn not merely according to its limiting surface. For if there were a finite luminous body extended in the aforesaid way, and if it would act only according to its terminal surface indivisible as to depth, it now follows that a finite surface–light would produce, in the neighboring infinite medium outside, as light which was absolutely infinite, and that the effect of a finite agent would be absolutely infinite.

5.5 Oresme. Area representing distance

[ref. 16]

13 That is, Part III, chapter iv.
5.6  Cardinal Nicholas of Cusa, c. 1450

We wish to find the relation between the area of a circle and its circumference. For simplicity we suppose that the radius of the circle is 1. Now, the circle can be thought of as composed of infinitely many straight–line segments, all equal to each other and infinitely short. The circle is then the sum of infinitesimal triangles, all of which have altitude 1. For a triangle the area is half the base times the altitude. Therefore the sum of the areas of the triangles is half the sum of the bases. But the sum of the areas of the triangles is the area of the circle, and the sum of the bases of the triangles is its circumference. Therefore the area of the circle of radius 1 is equal to one half its circumference.\[14\]

\[14\]  Argument attributed to Nicholas by Davis and Hersch. See [DH], pages 238-9.
Chapter 6

Before Newton and Leibniz

6.1 Simon Stevin, *Elements of the Art of Weighing*, 1585

THEOREM II. PROPOSITION II

The center of gravity of any triangle is in the line drawn from the vertex to the middle point of the opposite side.

Supposition. Let $ABC$ be a triangle of any form, in which from the angle $A$ to $D$, the middle point of the side $BC$, there is drawn the line $AD$.

What is required to prove. We have to prove that the center of gravity of the triangle is in the line $AD$.

Preliminary. Let us draw $EF$, $GH$, $IK$ parallel to $BC$, intersecting $AD$ in $L$, $M$, $N$; after that $EO$, $GP$, $IQ$, $KR$, $HS$, $FT$, parallel to $AD$.

Proof. Since $EF$ is parallel to $BC$, and $EO$, $FT$ to $LD$, $EFTO$ will be a parallelogram, in which $EL$ is equal to $LF$, also to $OD$ and $DT$, in consequence of which the center of gravity of the quadrilateral $EFTO$ is in $DL$, by the first proposition of this book.\footnote{Beginhelen der Weegconst Elements of the art of weighing, Leiden, 1585. From Struik. Pages 189–191.}\footnote{Theorem I, Proposition I: “The geometrical center of any plane figure is also its center of gravity.”} And for the same reason the center of gravity of the parallelogram $GHSP$ will be in $LM$, and of $IKRQ$ in $MN$; and consequently the center of gravity of the figure $IKRHSFTOEPGQ$, composed of the aforesaid three quadrilaterals, will be in the line $ND$ or $AD$. Now as here...
three quadrilaterals have been inscribed in the triangle, so an infinite number of such quadrilaterals can be inscribed therein, and the center of gravity of the inscribed figure will always be (for the reasons mentioned above) in the line \( AD \). But the more such quadrilaterals there are, the less the triangle \( ABC \) will differ from the inscribed figure of the quadrilaterals. For if we draw lines parallel to \( BC \) through the middle points of \( AN, NM, ML, LD \), the difference of the last figure will be exactly half of the difference of the preceding figure[4]. We can therefore, by infinite approximation, place within the triangle a figure such that the difference between the latter and the triangle shall be less than any given plane figure, however small. From which it follows that, taking \( AD \) to be the center line of gravity[5], the apparent weight of the part \( ADC \) will differ less from the apparent weight of the part \( ADB \) than any plane figure that might be given, however small, from which I argue as follows[6].

A. Beside any different apparent gravities there may be placed a gravity less than their difference;

0. Beside the present apparent gravities \( ADC \) and \( ADB \) there cannot be placed any gravity less than their difference;

0. Therefore the present apparent gravities \( ADC \) and \( ADB \) do not differ.

Therefore \( AD \) is the center line of gravity, and consequently the center of gravity of the triangle \( ABC \) is in it.

Conclusion. The center of gravity of any triangle therefore is in the line drawn from the vertex to the middle point of the opposite side, which we had to prove.

Problem I, Proposition III. Given a triangle: to find its center of gravity.

Supposition. Let \( ABC \) be a triangle.

What is required to find. We have to find its center of gravity.

3 It is obviously assumed that the side \( AB \) is divided into \( n \) equal segments (in the figure \( n = 4 \)). The difference between the area \( \Delta \) of the triangle \( ABC \) and that of the figure consisting of \((n - 1)\) parallelograms is \( \Delta/n \).

4 The statement that \( AD \) is the center line of gravity seems to mean that \( AD \) is the vertical through the point of suspension of the triangle at rest and hence, by the rule of statics quoted by Stevin earlier in the book (Book I, Prop. 6: The center of gravity of a hanging solid is always in its center line of gravity), the center of gravity is in \( AD \). (Notes 2 and 3 are based on footnotes in the Principal works, I).

5 Stevin here uses the form of the syllogism known in ancient logic as CAMESTRES (vowels AEE, A universal affirmation, as all P are Q, E universal negation, as no P are Q). He uses this formulation repeatedly (see Principal works, I, 143, note 2).

The reasoning amounts to this: When we know that the difference of two quantities \( A \) and \( B \) is smaller than a quantity that can be taken as small as we like, then \( A = B \). The reductio ad absurdum, typical of the Greeks, is replaced by a syllogism.

It has been justly observed that Stevin’s way of reasoning constitutes an important step in the evolution of the limit concept; see H. Bosmans, “Sur quelques exemples de la théorie des limites chez Simon Stevin,” Annales de la Societe Scientifique de Bruxelles 37 (1913), 171-199; “L’Analyse infinitésimale chez Simon Stevin,” Mathesis 37 (1923), 12-18, 55-62, 105-109, summarized in sec. V of Bosmans, “Le mathematicien belge Simon Stevin de Bruges,” Periodico de mathematiche (ser. 4) 6 (1926), 231-261.
Construction. There shall be drawn from $A$ to the middle point of $BC$ the line $AD$, likewise from $C$ to the middle point of $AB$ the line $CE$, intersecting $AD$ in $F$. I say that $F$ is the required center of gravity.

Proof. The center of gravity of the triangle $ABC$ is in the line $AD$, and also in $CE$, by the second proposition. It is therefore $F$, which we had to prove.

Conclusion. Given therefore a triangle, we have found its center of gravity, as required.

6.2 Quote of Valerio, 1604

If a quantity, either greater or smaller than a first quantity, has had a proportion to a quantity greater or smaller than a second quantity, with an excess or defect smaller than any arbitrary quantity \( \textit{excessu, vel defectu quantacumque proposita} \), then the first quantity will have to the second the same proportion.

6.3 Johannes Kepler, \textit{New Solid Geometry of Wine Barrels}. 1615

Part I. The Solid Geometry of Regular Bodies

Theorem I. We first need the knowledge of the ratio between circumference and diameter. Archimedes taught:

The ratio of circumference to diameters is about $22:7$. To prove it we use figures inscribed in and circumscribed about the circle. Since there is an infinite number of such figures, we shall, for the sake of convenience, use the hexagon. FIG. Let a regular hexagon $CDB$ be inscribed in the circle; let its angles be $C$, $D$, $B$, its side $DB$ and $F$ the point of intersection of the two tangents at $B$ and $D$ respectively. The line $AF$ connects the center $A$ with $F$, and intersects the line $DB$ at $G$, the curve $DB$ at $E$. But as $DGB$ is a straight line, it is the shortest distance between $D$ and $B$.

$DEB$, on the other hand, being a curve, is not the shortest distance between $D$ and $B$. Hence $DEB$ is longer than $DGB$. On the other hand, $BF$ is tangent

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6 Luca Valerio (1552-1618), \textit{De centra gravitatis solidorum} (Rome, 1604; 2nd ed., Bologna, 1659):
7 Linz, 1615. Taken from Struik. Pages 192–197.
to the circle and therefore all parts of the curve $EB$ are between $FB$ and $GB$; therefore, if $EB$ were straight, it would altogether be shorter than $FB$. For $AEB, FEB$ are equivalent to a right angle, and, as $EFB$ is an acute angle, $EB$, opposite the smaller angle $EFB$, must be smaller than $FB$, since this is opposite the larger angle. And we can consider $EB$ a straight line, because in the course of the proof the circle is cut into very small arcs, which appear to be equal to straight lines.

Now since, as can be observed, the curve $DEB$ is contained in the triangle $DBF$, it must be smaller than the lines $DF, FB$, since it bends toward the angle $DFB$, and still has not the slightest part outside the lines $DF, FB$; but the containing, according to common sense, is greater than the contained. This would be different, were the curve $DEB$ winding and irregular.

But as $DB$ is a side of the inscribed hexagon, and $DF, FB$ are two halves of the circumscribed hexagon, arc $DEB$ must be a sixth of the circle, since it was greater than $DB$ and smaller than $DF, FB$; $6DB$ is smaller than the circumference of the circle and $12DF$ (or $FB$) is greater than the circumference.

But the side $DB$ to the regular hexagon is equal to the radius $AB$. Therefore 6 radii $AB$, that is, three diameters $CB$ or (if the diameter is divided by 7) $\frac{21}{7}CB$ are shorter than the circumference.

And again, since $DG, GB$ are equal, $GB$ is half of $AB$. The square of $AB$, however, is equal to the sum of the squares of $AB$ and $GB$ and is the quadruple of the square of $GB$. Therefore the square of $AG$ is three times the square of $GB$. The ratio therefore of the squares of $AB$ and $AG$ is $\frac{4}{3}$ of the lines, therefore the ratio $AB : AG$ is $\sqrt{\frac{4}{3}}$, that is, the ratio of the numbers 100,000 : 86,603.

But as $AG : AB :: GB : BF$, then also $BF : GB$ is $\sqrt{\frac{4}{3}}$ and as $GB$ is half of $AB$, for example, 50,000, $BF$ must have about 57,737 such parts. Twelvefold this total number there will be greater than the circumference of the circle. Computation gives the number 477,974 for those circles which have 200,000, for diameter. And those of diameter 7 have for twelve times $BF$ the value $24 - \frac{1}{10}$. But this number is greater than the circumference itself; on the other hand the number 21 is smaller than the said circumference. And it is obvious that the curve $BE$ is nearer to $BG$ than the line $BF$. The circumference therefore is nearer the number 21 than $24 - \frac{1}{10}$. We suppose it differs by 1 from 21, from the other by $2 - \frac{1}{10}$, and that it therefore doubtless is 22. This, however, Archimedes shows much more accurately by means of multisided figures of 12, 24, 48 sides; there it also becomes apparent how little the difference of the circumference from 22 is. Adranus Romanus proved by the same method that when the diameter is divided into 20,000,000,000,000,000 parts, then about 62,831,853,071,795,862 of those parts make up the circumference.

Remark. Of the three conical lines, which are called parabola, hyperbola, and ellipse, the ellipse is similar to the circle, and I showed in the Commentary on the motion of Mars that the ratio of length of the elliptic line to the arithmetic

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8 Ideae mathematicae, Louvain, 1593
mean of its two diameters (which are called the right and transversal axes) is
about equal to $22 : 7$.

Theorem II. The area of a circle compared with the area of the square erected
on the diameter has about the ratio $11 : 14$.

Archimedes uses an indirect proof in which he concludes that if the area
exceeds this ratio it is too large. The meaning of it seems to be this.

The circumference of the circle $BG$ has as many parts as points, namely, an
infinite number; each of these can be regarded as the base of an isosceles
triangle with equal sides $AB$, so that there are an infinite number of
triangles in the area of the circle.

They all converge with their vertices
in the center $A$. We now straighten the circumference of circle $BG$ out into
the line $BC$, equal to it. The straight line $BC$, arranged on next to the other. Let
$BF$ be one of these bases, and and $CE$ any other, equal to it, and let the oints
$F$, $E$, $C$ be connected with $A$. Since there are as many triangles $ABF$, $AEC$
over the line $BC$ as there are sectors in teh area of the circle, and the bases
$BF$, $EC$ are equal, and all have the altitude $BA$ in common (which is also one
of the sectors), the triangles $EAC$, $BAF$ will be equal, and equal to one of the
circle sectors. As they all have their bases on $BC$, the triangle $BAC$, consisting
of all those triangles, will be equal to all the sectors of the circle and therefore
equal to the area of the circle which consists of all of them. This is equivalent
to Archimedes’ conclusion by means of an absurdity.

If now we divide $BC$ in half at $H$, then $ABHD$ forms a parallelogram. Let
$DH$ intersect $AC$ in $I$. This parallelogram is equal to the circle in area. Indeed,
$CB$ is to it half $CH$ as $AB$ (that is, $DH$) is to its half $IH$. Therefore $IH = ID$
and $HC = AD$ (equal to $BH$). The angles at $I$ are equal, and those at $D$ and
$H$ are right angles. The triangle $ICH$, which is outside the parallelogram, is
equal to triangle $IAD$ by which the parallelogram exceeds the trapezoid $AIHB$.

If now the diameter $GB$ is 7 parts, then its square will be 49. And since
the circumference consists of 22 such parts – hence also $BC$ – its half $BH$ will
consist of 11, hardly more or less. Multiply it by the semidiameter $3\frac{1}{2}$, which
is $AB$, and we get for the rectangle $AH$ $38\frac{1}{2}$. Therefore, if the square of the
diameter is 49, the area of the cirecle is as twice 49 or 98 to 77. Dividing by 7
we obtain 14 to 11. Q.E.D.

Corollary 1. The area of the sector of a circle (consisting of straight lines
from the center intersecting the arc) is equal to the rectangle over the radius
and half the arc.

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Theorem XVIII. Any ring with circular or elliptic cross section is equal to a
cylinder whose altitude equals the length of the circumference which the center
of the rotated figure describes, and whose base is the same as the cross section
of the ring.
By cross section is meant the intersection of a place through the center of the ring-shaped space and perpendicular to the ring-shaped surface. The proof of this theorem follows partly from theorem XVI and can be established by the same means by which Archimedes taught us as the principles of solid geometry.

Indeed, if we cut the ring $GCD$ from its center $A$ into an infinite number of very thin disks, any one of them will be the thinner toward the center $A$, the nearer its part, such as $E$, lies to the center $A$ than to $F$ and the normal through $F$ erected in the intersecting plane to the line $ED$. It also will be the thicker the nearer it is to the point $D$. At such two extreme points, such as $D$ and $E$, the sum of the two thicknesses will be twice the one in the middle of the disk.

This consideration would not be valid if the parts at $E$ and $D$ of the disk on either side of the circumference $FG$ and the perpendiculars through $F$ and $G$ were not equal and equally situated.

**Corollary.** This mode of measuring is valid for circular and for elliptical rings as well, high, narrow, or reclining, for open and closed rings alike, as indeed even for all rings whatever shape their cross section may have (instead of the circle $ED$) – so long as in the plane through $AD$ perpendicular to the ring the parts on either side of $F$ are equal and equally situated. We shall explore this in the case of a square section. Let the ring be of square shape and assume the square to be on $ED$. This ring can also be measured in another way. For it is the outer part of a cylinder whose base is a circle with $AD$ as radius and whose height is $DE$. From this cylinder, according to Theorem XVI, the middle part has to be subtracted, that is, the cylinder whose base is the circle of radius $AE$ and whose height is $ED$. The product, therefore, of $ED$ and the circular area $AD$ minus the circular area $AE$ is equal to the volume of the ring with a square as cross section. And if $ED$ is multiplied by the difference of the squares of $AD$ and $AE$, then the ratio of this body to the fourth part of the ring would be as the square to the circle, therefore as 14 to 11. Let $AE$ be equal to 2, $AD$ equal to 4, then its square is 16; but the square of $AE$ is 4, therefore the difference of the squares is 12; this number multiplied by the altitude 2 gives the volume as 24, of which the quadruple is 96. Since 14 is to 11 as 96 : $\frac{75}{2}$, the volume of the square ring is $75\frac{3}{2}$. This is according to the computation of Theorem XVI. And according to the preceding method, if $AF$ is 3, $FG$ is 6. Since 7 is to 22 as 6 is to 19 minus $\frac{1}{7}$, this therefore will be the length of the circumference $FG$, the altitude of the cylinder. And since $ED = 2$, its square is 4. To obtain the base of the cylinder, multiply therefore 4 by ($19 - \frac{1}{7}$). In this way also we see the truth of the theorem.

**Theorem XIX and Analogy.** A closed ring is equal to a cylinder whose base is the circle of the cross section and whose height equals the circumference of the circle described by its center.

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9 Theorem XVI deals with the ratio of conical segments of the same height and difference bases.
As this method is valid for every ring, whatever the ratio of $AE$ and $AF$ may be, therefore it also holds for a closed ring, in which the center $F$ of the circle $ED$ describes the circle $FG$, where $FG$ is equal to the rotated $AD$ itself. This is because such a closed ring is intersected from $A$ in disks that have no thickness at $A$ and at $D$ twice the thickness of that at $F$. Hence the circle through $D$ is twice that through $F$.

**Corollary.** The cylindric body that is created by rotation of $MIKN$, the four sided figure of straight and curved lines, is according to the same consideration equal to a column with this figure as base and the length of the circle $FG$ as height. But the outer fringe $IKD$ that surrounds the cylindric body – as a wooden hoop surrounds a barrel – clearly does not yield to this theorem, and must be computed by other means.

**Analogy.** Moreover, this method is valid for all cylindric bodies or parts of apples (or figs), no matter how slender, until $I, K$ coincide with $M, N$, which happens in the formation of the sphere, where instead of the two lines $MN$ and $IK$ there exists only one, namely, $BC$. For this body the demonstration anduse of this theorem fail for the first time.

**Corollary.** The ratio of the sphere to the closed ring created by the same circle is $7$ to $33$, since one–third of the radius multiplied by four times the area of the greatest circle, or two– thirds of the diameter multiplied by the area of the greatest circle, produce a cylinder equal to the sphere. And a cylinder equal to the closed ring has the same ase, and its altitude is the circumference [formed by the center]. Therefore as the circumference is to two– thirds of the diameter, that is $33 : 7$ [i.e., $3\pi : 2$], so is the ring to the sphere. [ref 20]

### 6.4 Bonaventura Cavalieri

#### 6.4.1 A Certain Method for the Development of a New Geometry of Continuous Indivisibles

The Theorem. 1 If between the same parallels any two plane figures are constructed, and if in them, any straight lines being drawn equidistant from the parallels, the included portions of any one of these lines are equal, the plane figures are also equal to one another; and if between the same parallel planes any solid figures are constructed, and if in them, any planes being drawn equidistant from the parallel planes, the included plane figures out of any one of the planes so drawn are equal, the solid figures are likewise equal to one another. [ref 21, 21.1]

The figures so compared let us call analogues, the solid as well as the plane...

The Proof. Let any two plane figures $ABC$ and $XYZ$ be constructed between the same parallels $PQ$, $RS$; and let $DN$, $OU$, be drawn parallel to the aforesaid $PQ$, $RS$; and let the portions, for example of $DN$, included in the figures, namely $JK$, $LM$, be equal to each other; and again, in the line $OU$, let the portions $EF$, $GH$, taken together (for the figure $ABC$, for example, may be hollow within, according to the contour of $FfG$), be likewise equal to $TV$; and let this happen in all the other lines equidistant from $PQ$. I say that the figures, $ABC$, $XYZ$, are equal to each other.

Let either, then of the two figures $ABC$, $XYZ$ be taken, for example $ABC$ itself, with the portions of the parallels $PQ$, $RS$ coterminous with it, namely the portions $PA$, $RB$, and let it be superposed upon the other figure $XYZ$, but so that the lines $PA$, $RB$ may fall upon $AQ$, $CS$; then either the whole figure $ABC$ coincides with the whole figure $XYZ$ (and thus, since they coincide with each other they are equal), or not; yet let there be some part which will coincide with some part, as $XMCYThL$, part of the figure $ABC$, with $XMCYThL$ part of the figure $XYZ$.

It is manifest, moreover, if the superposition of the figures is effected in such a way that portions of the parallels $PQ$, $RS$ coterminous with our two figures are mutually superposed, that whatever straight lines (included in the figures) are in line remain in line; as, for example, since $EF$, $GH$ are in line with $TV$, when the aforesaid superposition is made they will remain in line (namely $EFTh$ in line with $TV$), for the distance of those lines $EF$, $GH$ will always remain in line with $TV$, which is clearly apparent not only for this but for all other lines parallel to $PQ$ in either figure.

In the case where part of one figure (as $ABC$) coincides of necessity with part of the figure $XYZ$, and not with the whole, granting that the superposition be made by such a rule as has been told, the demonstration will be as follows. For since when any parallels are drawn to $PQ$, the portions of them, included in the figures, which were in line, will still remain in line after superposition, and moreover since they were by hypothesis equal before superposition, therefore, after superposition the portions included in the figures will likewise be equal – as for example, $EF$, $TH$ taken together will be equal to $TV$ – therefore, if $EF$, $TH$ do not coincide with the whole of $TV$, then, one part [of one] coinciding with one part [of the other], as $TH$ with $TH$ itself, $EF$ will be equal to $HV$, $EH$ being in the residuum of the figure $ABC$ which is superposed, and $HV$ in the figure $XYZ$ upon which the other is superposed. In the same way we shall show that to any line whatever parallel to $PQ$, and included in the residuum of the superposed figure $ABC$ (which may be $LBYTF$) corresponds an equal
straight line, in line [with the former], which will be in the residuum of the figure \textit{XYZ} on which \textit{ABC} is superposed; therefore, the superposition being made by this rule, when anything of the superposed figure is left over and does not fall upon the figure, it must be that something of the other figure must also be left over, and have nothing superposed upon it.

Since, moreover, to each of the straight lines parallel to \textit{PQ} and included in the residuum (or residua, for there may be several residual figures) of the superposed figure \textit{ABC} (or \textit{XBC}) there corresponds another straight line, in line [with the first] and included in the residuum (or residua) of the figure \textit{XYZ}, it is manifest that these residual figures, or their aggregates, are between the same parallels; so since the residual figure \textit{LBYTF} is between the parallels \textit{DN}, \textit{RS}, likewise the residual figure (or aggregate of residual figures) of the figure \textit{XYZ} (because it has the frusta \textit{Thg, MCZ}) will be between the same parallels \textit{DN}, \textit{RS}. For if it did not extend both ways to the parallels \textit{DN}, \textit{RS}, as for example if it extended up to \textit{DN}, but not down to \textit{RS}, only as far as \textit{OU}, then to the straight lines included in the frustum \textit{EBYfF}, and parallel to \textit{PQ}, there would not be found in the residuum of the figure \textit{XYZ} (or in the aggregate of the residua) other corresponding lines as has been proved to be unavoidable. Therefore these residua, or their aggregates, are between the same parallels; and the portions of the lines parallel to \textit{PQ}, \textit{RS}, included therein, are equal, as we have shown above; therefore the residua are subject to the same condition as has been assumed for \textit{ABC}, \textit{XYZ}; that is, they are analogous.

So let the residua be now superposed, but so that the parallels \textit{KL}, \textit{CY} may fall upon the parallels \textit{LN}, \textit{YS}, and the part \textit{VBZ} of the frustum \textit{LBYTF} may coincide with the part \textit{VBZ} of the frustum \textit{MCZ}; then we shall show, as above, that as long as there is found a residuum of one, there will be found also a residuum of the other, and these residua, or aggregates of residua, will be found within the same parallels. Let \textit{LVZYGF} be a residuum belonging to the figure \textit{ABC}; and let \textit{MCBV, Thg}, be residua belonging to the figure \textit{XYZ}, whose aggregate is between the same parallels as the residuum \textit{LVZYGF}, that is, between \textit{DN}, \textit{RS}. If we now superpose these residua again, but so that the parallels between which they lie be always superposed respectively, and this is supposed to be done continually, until the whole figure \textit{ABC} shall have been superposed, I say the whole of it must coincide with \textit{XYZ}; otherwise if there were any residuum of the figure \textit{XYZ}, upon which nothing is superposed, there would also be some residuum of the figure \textit{ABC} which would not have been superposed, as we have shown above to be unavoidable; but it is granted that the whole of \textit{ABC} is superposed upon \textit{XYZ}, therefore they are so superposed upon each other that there are no residua of either, therefore they are so superposed that they coincide, therefore the figures \textit{ABC}, \textit{XYZ} are equal to each other.

Now in the same diagram let \textit{ABC, XYZ} be any two solid figures constructed between the same parallel planes \textit{PQ}, \textit{RS}; and let \textit{DN}, \textit{OU} be any planes drawn equidistant from the planes previously spoken of; and let the figures that lie in the same plane and that are included in the solids be equal to each other always; as \textit{JK} equal to \textit{LM}, and \textit{EF, GH}, taken together (for a
solid figure, for example \(ABC\), may be hollow in any way within, according to the surface \(FfGg\), equal to \(TV\). I say that these solid figures are equal to each other.

For if we superpose the solid \(ABC\), with the portions \(PA, RC\) of the planes \(PQ, RS\), coterminal with it, upon the solid \(XYZ\), in such a way that the plane \(PA\) be on the plane \(PQ\), and the plane \(RC\) on the plane \(RS\), we shall show (as we did above about the portions of the lines parallel to \(PQ\) included in the plane figures \(ABC, XYZ\)) that the figures included in the solids and lying in the same plane will also after superposition remain in the same plane; and therefore thus far the figures included in the superposed solids are equal – and parallel to \(PQ, RS\).

Then unless the entire solid coincides with the other solid entire in the first superposition, residual solids will remain, or solids composed of residua, in either solid, which will not be superposed upon each other. Since for example the figures \(E'F', TH'\) are equal to the figure \(TV\), then when the common figure \(TH'\) is taken away, the remaining figure \(E'F'\) will be equal to the remaining figure \(H'V\); and this will happen in any plane whatever parallel to \(PQ\) and meeting the solids \(ABC, XYZ\). Therefore whenever we have a residuum of one solid, we shall always have a residuum of the other also; and it will be evident, according to the method applied in the former part of this Proposition in the case of plane figures, that the residua of the solids, or the aggregates of residua, will always be between the same parallel planes (as the residua \(LB'YT', MC'Z\), \(Thg\) are between the same parallels \(DN, RS\)) and will be analogues.

Now if these residua be superposed again, so that the plane \(DL\) we be places on the plane \(LN\), and \(RY\) on \(YS\), and this is understood to be done continually, until \(ABC\), which is being superposed, is entirely taken, the entire solid \(ABC\) will finally coincide with the entire solid \(XYZ\). For when the entire solid \(BC\) is superposed upon \(XYZ\), unless they coincided there were be some residuum of one, as of the solid \(XYZ\), therefore also some residuum of the solid \(XB'C'\) or \(ABC\), and this residuum would not be superposed; which is absurd, for it is already assumed that the entire solid \(ABC\) is superposed on \(XYZ\). Therefore there will not be any residuum in these solids; therefore they will coincide; therefore the solid figures spoken of, \(ABC, XYZ\), will be equal to each other, which was to be proved of them.

6.4.2 *Geometria Indivisibilium Continuorum Nova Quadam Tatione Promota*, 1635

Book II, Definition I: If through opposite tangents to any given plane figure there are drawn two planes parallel to each other, either at right angles or inclined to the plane of the given figure, and produced indefinitely, one of which is moved towards the other, always remaining parallel until it coincides with it, then the single lines, which in the motion as a whole are the intersections of the moving

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11 From Stendal, pages 62–65.
plane and the given figure, collected together are called: All the lines of the figure, taken one of them as *regula*.

...  

Book II, Theorem III: Plane figures have the same ratio to each other, as that of all their lines taken from whatever *regula*.

...  

Theorem IV. Proposition IV.

Suppose two plane figures, or solids, are constructed to the same altitude; moreover having taken straight lines in the planes, or planes in the solids, parallel to each other in whatever way, with respect to which the aforesaid altitude is taken, if it is found that segments of the taken lines intercepted in the plane figures, or portions of the taken planes intercepted in the solids, are proportional quantities, always in the same way in each figure, then the said figures will be to each other as any one of the former to the latter corresponding to it in the other figure.

First suppose the two plane figures constructed to the same altitude are $CAM$, $CME$, in which there are understood to be taken any two straight lines parallel to each other, $AE$, $BD$, with respect to which the common altitude is understood to be taken; moreover let there be intercepted segments $AM$, $BR$, inside figure $CAM$, and $ME$, $RD$, inside figure $CME$; and suppose it is found that $AM$ to $ME$ is as $BR$ to $RD$. I say that figure $CAM$ to figure $MCE$, will be as $AM$ to $ME$, or $BR$ to $RD$, for since $BD$, $AE$, however taken, are parallel to each other, it is clear that any one of those which are said to be all the lines of figure $CAM$, taken from either $AM$, $BR$ or *regula* to that which lies opposite to it in figure $CME$, will be as $BR$ to $RD$, or as $AM$ to $ME$; therefore as $AM$ to $ME$, that is, as one of the former to one of the latter so will be all of the former, namely all the lines of figure $CAM$, from *regula* $AM$, to all the latter, that is, to all the lines of the figure $CME$, from *regula* $ME$. The indefinite number $n.$ of all the former, or latter, is here the same for both, whatever it is (for here the figures are to the same altitude, and to any supposed former line in the figure $CAM$ there corresponds a latter constructed opposite to it in the other figure) so it cannot be but that all the lines of figure $CAM$ are comparable to all the lines of figure $CME$, since they have a ratio to them, as has been shown. And therefore all the lines of figure $CAM$, from *regula* $AM$, to all the lines of figure $CME$, from *regula* $ME$, will be as $AM$ to $ME$: but as all the lines of figure $CAM$ to all the lines of figure $CME$, so is figure $CAM$ to figure $CME$; therefore figure $CAM$ to figure $CME$ will be as $BR$ to $RD$, or $AM$ to $ME$, which it was required to show for plane figures.

But if we assume $CAM$, $CM$, to be solid figures, and instead of the lines $AM$, $BR$, $ME$, $RD$, we understand intercepted planes parallel to each other inside the figures $CAM$, $CME$, and so constructed that planes $AM$, $ME$, lie
in the same plane, just as do the planes $BR$, $RD$, with respect to which the aforesaid altitude is again understood to be taken, the proceeding by the same method we show that all the planes of figure $CAM$ to all the planes of figure $CME$, that is, the solid figure $CAM$ to the solid figure $CME$, are as the plane $BR$ to the plane $RD$, or as the plane $AM$ to the plane $ME$, which it was required to show also for solid figures.

6.5 Comparison of Kepler and Cavalieri’s methods.

Good source of discussion questions[ref 21.b]

6.6 Rene Descartes. Geometry. 1637

'most useful and most general problem in geometry I know’. Maybe include some of the curves? 1637. [ref 22]

6.7 Pierre de Fermat

6.7.1 Quadrature, circa 1658

ON THE TRANSFORMATION AND EMENDATION OF EQUATIONS OF PLACE\footnote{Fermat, \textit{Varia Opera}, 1679, pages 44-46. from Stedal. Stedal pages 78–84}

in order to compare curves in various ways with each other, or with straight lines

TO WHICH IS ADJOINED

THE USED OF GEOMETRIC PROGRESSIONS

\textit{in the quadrature of infinite parabolas or hyperbolas}

Archimedes made use of geometric progressions only for the quadrature of one parabola. In the remaining comparisons of heterogeneous quantities he restricted himself merely to arithmetic progressions. Whether because he found geometric progressions less appropriate? Or because the required method with the particular progression used for squaring the first parabola could scarcely be extended to the others? I have certainly recognized, and proved, progressions of this kind very productive for quadratures, and my discovery, by which one may square both parabolas and hyperbolas by exactly the same method, I by no means unwillingly communicate to more modern geometers.

I attribute to geometric progressions only what is very well known, on which this whole method is based.
The theorem is this: Given any geometric progression whose terms decrease infinitely, as the difference of two [consecutive] terms constituting the progression is to the smaller of them, so is the greatest term of the progression to the rest taken infinitely.

This established, there is proposed first the quadrature of hyperbolas. Moreover we defined hyperbolas as infinite curves of various kinds, like $DSEF$, of which this is a property, that having placed at any given angle $RAC$ its asymptotes, $AR$, $AC$, extended infinitely if one pleases but not cut by the curve, and taking whatever straight lines, $GE$, $HI$, $ON$, $MP$, $RS$, etc. parallel to one asymptote, we suppose that a certain power of the line $AH$ to the same power of the line $AG$ is as a power of the line $GE$, whether the same or different from the preceding one, to that same power of the line $HI$; moreover we understand the powers to be no only squares, cubes, square-squares, etc. of which the exponents are 2, 3, 4 etc. but also simple lines, whose power is one. I say, therefore, that all hyperbolas of this kind indefinitely, with one exception, which is that of Apollonius, or the first, can be squared with the help of the same an always applicable method of geometric progressions.

Let there be, if one likes, a hyperbola of which it is the property that the square of the lines $HA$ to the square of the lines $AG$ is always as the line $GE$ to the line $HI$, and that the square of $OA$ to the square of $AH$ is as the line $HI$ to the line $ON$, et. I say that the infinite space whose base is $GE$, and with the curves $ES$ for one side, but for the other the infinite asymptote $GOR$, is equal to a given rectilinear space. It is supposed that the terms of a geometric progression can be extended infinitely, of which the first is $AG$, the second $AH$, the third $AO$, etc. infinitely, and these approach each other by approximation as closely as is needed, so that by the method of Archimedes the parallelogram made by $GE$ and $GH$ adequates, as Diophantus says, to the irregular four-sided shape $GHE$, or very nearly equals.

$$GE \times GH$$

Likewise, the first of the straight line intervals of the progression $GH$, $HO$, $OM$, and so on, are similarly very nearly equal amongst themselves, so that we can conveniently use the method of exhaustion, and by Archimedean circumscriptions adn inscriptions the ratio to be demonstrated can be established, which it is sufficient to have shown once, nor do I wish to repeat or insist more often on a method already sufficiently known to any geometer.

This said, since $AH$ to $AO$ is as $AG$ to $AH$, so also will $AO$ to $AM$ be as $AG$ to $AH$. So also will be the interval $GH$ to $HO$, and the interval $HO$ to $HM$, etc. Moreover the parallelogram made by $EG$ and $GH$ will be to the parallelogram made by $HI$ and $HO$, as the parallelogram made by $HI$ and $HO$ to the parallelogram made by $NO$ and $OM$, for the ratio of the parallelogram made by $GE$ and $GH$ to the parallelogram made by $HI$ and $HO$ is composed from the ratio of the line $GE$ to the line $HI$, and from the ratios of the line $GH$ to the line $HO$; and as $GH$ is to $HO$, so is $AG$ to $AH$, as we have shown. Therefore the ratio of the parallelogram made by $EG$ and $GH$ to the parallelogram made
CHAPTER 6. BEFORE NEWTON AND LEIBNIZ

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6.7.2 Derivatives and Applications, circa 1638

(1) On a Method for the Evaluation of Maxima and Minima

The whole theory of evaluation of maxima and minima presupposes two unknown quantities and the following rule:

Let \( a \) be any unknown of the problem (which is in one, two, or three dimensions, depending on the formulation of the problem). Let us indicate the maximum or minimum by \( a \) in terms which could be of any degree. We shall now replace the original unknown \( a \) by \( a + e \) and we shall express thus the

Struik: “This paper was sent by Fermat to Father Marin Mersenne, who forwarded it to Descartes. Descartes received it January, 1638. It became the subject of a polemic discussion between him and Fermat.” Reference Struik, pages 222-24.
maximum or minimum quantity in terms of \(a\) and \(e\) involving any degree. We shall adequate, to use Diophantus’ term, the two expressions of the maximum or minimum quantity and we shall take out their common terms. Now it will turn out that both sides contain terms in \(e\) or its powers. We shall divide all terms by \(e\), or by a higher power of \(e\), so that \(e\) will be completely removed from at least one of the terms. We suppress then all the terms in which \(e\) or one of its powers will still appear, and we shall equate the others; or, if one of the expressions vanishes, we shall equate, which is the same thing, the positive and negative terms. The solution of this last equation will yield the value of \(a\) which will lead to the maximum or minimum, by using again the original expression.

Here is an example:

*To divide the segment \(AC\) at \(E\) so that \(AE \times EC\) may be maximum.*

We write \(AC = b\); let \(a\) be one of the segments, so that the other will be \(b - a\), and the product, the maximum of which is to be found, will be \(ba - a^2\). Let now \(a + ea\) be the first segment of \(b\); the second will be \(b - a - e\), and the product of the segments, \(ba - a^2 + be - 2ae - e^2\); this must be adequated with the preceding: \(ba - a^2\). Suppressing common terms: \(be \sim 2ae + e\). Suppressing \(e\): \(b = 2a\). To solve the problem we must consequently take the half of \(b\).

We can hardly expect a more general method.

**ON THE TANGENTS OF CURVES**

We use the preceding method to find the tangent at a given point of a curve.

Let us consider, for example, the parabola \(BDN\) with vertex \(D\) and of diameter \(DC\); let \(B\) be a point on it at which the line \(BE\) is to be drawn tangent to the parabola and intersecting the diameter at \(E\).

We choose on the segment \(BE\) a point \(O\) at which we draw the ordinate \(OI\); also we construct the ordinate \(BC\) of the point \(B\). We have then: \(CD/DI > BC^2/OI^2\), since the point \(O\) is exterior to the parabola. But \(BC^2/OI^2 = CE^2/IE^2\), in view of the similarity of the triangles. Hence \(CD/DI > CE^2/IE^2\).

Now the point \(B\) is given, consequently the ordinate \(BC\), consequently the point \(C\), hence also \(CD\). Let \(CD = d\) be this given quantity. Put \(CE = a\) and \(CI = e\); we obtain

\[
\frac{d}{d - 3} > \frac{a^2}{a^2 + e^2 - 2ae}.
\]

Removing the fractions:

\[
da^2 + de^2 - 2dae > da^2 - a^2e.
\]

\footnote{Isaac Newton, 1671 “When a quantity is greatest or least, at that moment its fluxion neither increases nor decreases: for if it increases, that proves that it was less and will at once be greater than it now is, and conversely if it decreases. Therefore seek its fluxion and set it equal to nothing.” *On the method of Series and Fluxions*}
Let us then adequate, following the preceding method; by taking out the common terms we find:

\[ de^2 - 2dae \sim -a^2e, \]

or, which is the same,

\[ de^2 + a^2e \sim 2dae. \]

Let us divide all terms by \( e \): On taking out \( de \), there remains \( a^2 = 2da \), consequently \( a = 2d \).

Thus we have proved that \( CE \) is the double of \( CD \) — which is the result.

This method never fails and could be extended to a number of beautiful problems; with its aid, we have found the centers of gravity of figures bound by straight lines or curves, as well as those of solids, and a number of other results which we may treat elsewhere if we have time to do so.

I have previously discussed at length with M. de Roberval the quadrature of areas bounded by curves and straight lines as well as the ratio that the solids which they generate have to the cones of the same base and same height.

(2) CENTER OF GRAVITY OF PARABOLOID OF REVOLUTION, USING THE SAME METHOD

Let \( CBAV \) be a paraboloid of revolution, having or its axis \( IA \) and for its base a circle of diameter \( CIV \). Let us find its center of gravity by the same method which we applied for maxima and minima and for the tangents of curves; let us illustrate, with new examples and with new and brilliant applications of this method, how wrong those are who believe it may fail.

In order to carry out this analysis, we write \( IA = b \).

Let \( O \) be the center of gravity, and \( a \) the unknown length of the segment \( AO \); we intersect the axis \( IA \) by any plane \( BN \) and put \( IN = e \), so that \( NA = b - e \).

It is clear that in this figure and in similar ones (parabolas or paraboloids) the centers of gravity of segments cut off by parallels to the base divide the axis in a constant proportion (indeed, the argument of Archimedes can be extended by similar reasoning from the case of a parabola to all parabolas and paraboloids of revolution\(^{16}\)). Then the center of gravity of the segment of which \( NA \) is the axis and \( BN \) the radius of the base will divide \( AN \) in a point \( E \) such that

\[ NA/AE = IA/AO, \]

or, in formula, \( b/a = (b - e)/AE \).

The portion of the axis will then be \( AE = (ba - ae)/b \) and the interval between the two centers of gravity, \( OE = ae/b \).

15 \( f(x+e)-f(x) \approx 0 \). Then used to find tangent lines. [[sudden appearance of tangents!?]]

16 “All parabolas” means “parabolas of higher order”, \( y = kx^n \), \( n > 2 \)
Let $M$ be the center of gravity of the remaining part $CBRV$; it must necessarily fall between the points $N$ and $I$, inside the figure, in view of Archimedes’ postulate \textsuperscript{17} in On the Equilibrium of planes, since $CBRV$ is figure completely concave in the same direction.

But

$$\frac{\text{Part } CBRV}{\text{Part } BAR} = \frac{EO}{OM},$$

since $O$ is the center of gravity of the total figure $CAV$ and $E$ and $M$ are those of the parts.

Now in the paraboloid Archimedes,

$$\frac{\text{Part } CAV}{\text{Part } BAR} = \frac{IA^2}{NA^2} = \frac{b^2}{b^2 + e^2 - 2be},$$

hence, by dividing,

$$\frac{\text{Part } CBRV}{\text{Part } BAR} = \frac{2be - e^2}{b^2 + e^2 - 2be}.$$ 

But we have proved that

$$\frac{\text{Part } CBRV}{\text{Part } BAR} = \frac{EO}{OM}.$$ 

This section appears to be that which Fermat addressed to Mersenne for Roberval, with his letter of April 20, 1638.

Then in formulas,

$$\frac{2be - e^2}{b^2 + e^2 - 2be} = \frac{EO(= ae/b)}{OM};$$

hence

$$OM = \frac{b^2ae + ae^3 - bac^2}{2b^2e - be^2}.$$ 

From what has been established we see that the point $M$ falls between points $N$ and $I$; thus $OM < OI$; now, in formula, $OI = b - a$. The question is then prepared from our method, and we may write

$$b - a \sim \frac{b^2ae + ae^3 - bac^2}{2b^2e - be^2}.$$ 

Multiplying both sides by the denominator and dividing by $e$:

$$2b^3 - 2b^2a - b^2e + bae \sim b^2a + ae^2 - 2bae.$$ 

Since there are no common terms, let us take out all those in which $e$ occurs let us equate the others:

$$2b^32b^2a = b^2a,$$ hence $3a = 2b$.

\textsuperscript{17} According to Heath: “In any figure whose perimeter is concave in (one and) the same direction the center of gravity must be within the figure.”
Consequently,\[ \frac{IA}{AO} = \frac{3}{2}, \quad \text{and} \quad \frac{AO}{OI} = \frac{2}{1}, \]
and this was to be proved.

The same method applies to the centers of gravity of all parabolas ad infinitum as well as to those paraboloids of revolution. I do not have time to indicate, for example, how to look for the center of gravity in our paraboloid obtained by revolution about the ordinate\[^{16}\]it will be sufficient to say that, in this conoid, the center of gravity divides the axis into two segments in the ratio 11/5.

### 6.8 Galileo Galilei. Discourses. 1638

Infinites, tangents to parabolas, velocity and acceleration. mostly from Discourses, First Day. Lots!! of stuff here. Have to decide what to read. 1638. [ref 24]

### 6.9 Evangelista Torricelli

#### 6.9.1 geometric proof of sum of geometric series.

No reference. [ref 26]

#### 6.9.2 On the Acute Hyperbolid Solid, 1644

ON THE ACUTE HYPERBOLIC SOLID\[^{19}\]

Consider a hyperbola of which the asymptotes $AB$, $AC$ enclose a right angle. [Fig 1] If we rotate this figure about the axis $AB$, we create what we shall call an acute hyperbolic solid, which is infinitely long in the direction of $B$. Yet this solid is finite. It is clear that there are contained within this acute solid rectangles through the axis $AB$, such as $DEFG$. I claim that such a rectangle is equal to the square of the semiaxis of the hyperbola.

We draw from $A$, the center of the hyperbola, the semiaxis $AH$, which bisects the angle $BAC$. This gives us the rectangle $AIHC$, which is certainly a square (it is a rectangle and the angle at $A$ bisected by the axis $AH$.) Therefore the square of $AH$ is twice the square $AIHC$, or twice the rectangle $AF$, and therefore equal to the rectangle $DEFG$, as claimed\[^{20}\].

\[^{18}\] Here $ACI$ of figure 3 is rotated about $CI$.

\[^{19}\]Struik, pages 227-230., [ref 27]

\[^{20}\] $xy =$ constant, from Apollonius Conics, Book II, prop 12.
Lemma 2. All cylinders described within the acute hyperbolic solid and constructed about the common axis are isoperimetric (I always mean without their bases). Consider the acute solid with axis $AB$ [Fig 2] and visualize within it the arbitrary cylinders $CDEF$, $GHLI$, drawn about the common axis $AB$. The rectangles through the axes $CE$, $GL$ are equal and so the curved surfaces of the cylinders will be equal. Q.E.D.\(^{21}\)

Lemma 3. All isoperimetric cylinders (for instance, those that are drawn within the acute hyperbolic solid) are to each other as the diameters of their bases. Indeed, in Fig. 2, the rectangles $AE$, $AL$ are equal, hence $FE : IL = AI : AF$. The cylinder $CE$ has to cylinder $GL$ a ratio composed of $AF^2 : AI^2$ and $FE : IL$, or of $FA : IA$, or of $FA^2 : AI$ times $AF$. The cylinders $CE$, $GL$ are therefore to each other as $FA^2$ is to $AI$ times $AF$, and thus as line $FA$ is to line $AI$. Q.E.D.

Lemma 4. Let $ABC$ be an acute body with axis $BD$, $D$ the center of the hyperbola (where the asymptotes meet), and $DF$ the axis of the hyperbola. We construct sphere $AEFC$ with center $D$ and radius $DF$. this is the largest sphere with center $D$ that can be described in the acute body. We take an arbitrary cylinder contained in the acute body, say $GIHL$. I claim that the surface of cylinder $GH$ is one-fourth that of the sphere $AEFC$.

Indeed, since the rectangle $GH$ through the axis of the cylinder is equal to $DF^2$, hence to the circle $AEFC$, therefore this cylindrical surface $GIHL = \frac{1}{4}$ the surface of the sphere $AEFC$, of which the great circle $AEFG$ is also one-fourth.

Lemma 5. The surface of any cylinder $GIHL$ described in the acute solid (the surface without bases)is equal to the circle of radius $DF$, which is the semiaxis, or half the latus versum of the hyperbola, for this is proved in the demonstratio of the preceding lemma\(^{22}\).

Theorem. An acute hyperbolic solid, infinitely long, cut by a plane [perpendicular] to the axis, together with the cylinder of the same base, is equal to that right cylinder of which the base is the latus versum (that is, the axis) of the

\(^{21}\)uses Archimedes

\(^{22}\) Latus versum is what we’d call the real axis. Thinking of $AB$ as the $y$-axis, the equation of the hyperbola is $xy = a^2/2$, when the length of the latus versum is $2a$. 
hyperbola, and of which the altitude is equal to the radius of the basis of this acute body.

Consider a hyperbola of which the asymptotes \( AB, AC \) enclose a right angle. We draw from an arbitrary point \( D \) of the hyperbola a line \( DC \) parallel to \( AB \), and \( DP \) parallel to \( AC \). Then the whole figure is rotated about \( AB \) as an axis, so that the acute hyperbolic solid \( EBD \) is formed together with a cylinder \( FEDC \) with the same base. We extend \( BA \) to \( H \), so that \( AH \) is equal to the entire axis, that is, the latus versum of the hyperbola. And on the diameter \( AH \) we imagine a circle [in the plane] constructed perpendicularly to the asymptote \( AC \), and over the base \( AH \) we conceive a right cylinder \( ACGH \) of altitude \( AC \), which is the radius of the base of the acute solid. I claim that the whole body \( FEBDC \), though long without end, yet is equal to the cylinder \( ACGH \).

We select on the line \( AC \) an arbitrary point \( I \) and we form the cylindrical surface \( ONLI \) inscribed in the acute solid about the axis \( AB \), and likewise the circle \( IM \) on the cylinder \( ACGH \) parallel to the base \( AH \). Then we have, according to our lemma: (cylindrical surface \( ONLI \)) is to (circle \( IM \)) as (rectangle \( OL \) through the axis) is to (square of the radius of circle \( OM \)), hence as (rectangle \( OL \)) is to (square of the semiaxis of the hyperbola).

And this will always be true no matter where we take point \( I \). Hence all cylindrical surfaces together, that is, the acute solid \( EBD \) itself, plus the cylinder of the base \( FEDC \), will be equal to all the circles together, that is, to the cylinder \( ACGH \). Q.E.D.

### 6.10 de Saint-Vincent Gregoire. series.1640??

Trek 1.2 See Baron pg 135-47 Infinitesimal calculus. apparently nice stuff on geometric series and Zeno. [ref 28]

### 6.11 Cavalieri. integration of higher parabolas. 1647.

Very wordy - skip??[ref 29]

(PASTE IN SJC's STUFF)

[ref 30b]. Useful for derivatives.

6.13 Wallis. Arithmetic Infinitum. 1656

Uses modern notation, computes some integrals. Ref? Source? See wikipedia. [ref 32]

6.14 Wallis. On infinitesimals and ’smooth’ 1656

[ref 32bc]

6.15 John Wallis, On Indivisibles, 1656.

De Sectionibus conicis, 165623

Proposition 1.

Plane figures considered according to the method of indivisibiles.

I suppose at the start (according to the Geometria indivisibilium of Bonaventura Cavalieri) that any plane whatever consists, as it were, of an infinite number of parallel lines. Or rather (which I prefer) of an infinite number of parallelograms of equal altitude; of which indeed the altitude of a single one is \( \frac{1}{\infty} \) of the whole altitude, or an infinitely small divisor; (for let \( \infty \) denote an infinite numbers); and therefore the altitude of all of them at once is equal to the altitude of the figure.

Moreover, in whichever way the things ins explained (whether by infinitely many parallel lines, or by infinitely many parallelograms of equal altitude interposed between those infinitely many lines) it comes down to the same thing. For a parallelogram whose altitude is supposed infinitely small, that is, nothing (for an infinitely small quantity is just the same as no quantity) is scarcely other than a line. (In this at least they differ, that a line is here supposed to be dilatable, or at least to have a certain such thickness that by infinite multiplication it can acquire a definite altitude or latitude, namely as much as the altitude of the figure.) Therefore from now on (partly because strictly speaking this seems to have been the case in Cavalieri’s method of indivisibles, partly also so that we may deliberate with brevity) we sometimes call those infinitely tiny parts of figures (or, of infinitely tiny altitude) by the name of LINES rather than PARALLELOGRAMS, at least when we do not have to consider the determination of the altitude. Moreover, when we do have to take into consideration the

\(^{23}\)Stendal, pages 67-68., Ref[31]. stendall page 69
determination of the altitude (as sometimes happens) those tiny altitudes must have a ratio, so that infinitely multiplied they are assumed to become equal to the whole altitude of the figure.

6.16 Thomas Hobbes, on Wallis’ Infinitesimals, 1656.

from *Six lessons to the Professors of Mathematics*.

Therefore though your *Lemma* be true, and by me (Chap.13, Art.5) demonstrated; yet you did not know why it is true; which also appears most evidently in the first Proposition of your *Conique-sections*. Where first you have this, *That a Parallelogram whose Altitude is infinitely little, that is to say, none, is scarce anything else but a Line.* Is this the Language of Geometry? How do you determine this word *scarce*? The least Altitude, is Somewhat or Nothing. If Somewhat, then the first character of your Arithmetical Progression must not be zero; and consequently the first eighteen Propositions of this your *Arithmetica Infinitorum* are all naught. If Nothing, then your whole figure is without Altitude, and consequently your Understanding naught. Again, in the same Proposition, you say thus, *We will sometimes call those Parallelograms rather by the name of Lines then of Parallelograms, at least when there is no consideration of a determinate Altitude; But where there is a consideration of a determinate Altitude (which will happen sometimes) there that little Altitude shall be so far considered, as that being infinitely multiplied it may be equal to the Altitude of the whole Figure.* See here in what a confusion you are when you resist the truth. When you consider no determinate Altitude (that is, *no Quantity* of Altitude) then you say your Parallelogram shall be called a *Line*. But when the Altitude is determined (that is, when it is *Quantity* then you will call it a Parallelogram. Is not this the very same doctrine which you so much wonder at and reprehend in me, in your objection to my Eighth Chapter, and your word *considered* used as I use it? 'Tis very ugly in one that so bitterly reprehendeth a doctrine in another, to be driven upon the same himself by the force of truth when he thinks not on’t. Again, seeing you admit in any case, the *infinitely little* altitudes to be quantity, what need you this limitation of yours, *so far forth as that by multiplication they may be made equal to the Altitude of the whole figure*? May not the half, the third, the fourth, or the fifth part, etc. be made equal to the whole by multiplication? Why could you not have said plainly, *so far forth as that every one of those infinitly little Altitudes be not only something but an aliquot part [divisor] of the whole?* So you will have an *infinitely little* Altitude, that is to say a *Point*, to be both nothing and something and an *aliquot part*. And all this proceeds from not understanding the grounds of your Profession.

24 1656, page 46. From [31]
6.17 John Wallis, 1656, on Limits and Continuity

And whatever is so little or nothing in any kind, as that it cannot by Multiplication, become so great or greater than any proposed Quantity of that kind, is (as to that kind of Quantity,) None at all. ...[Euclid] takes this for a Foundation of his Process in such Cases: That those Magnitudes (or Quantities,) whose Difference may be proved to be Less than any Assignable are equal. For if unequal, their Difference, how small soever, may be so Multiplied, as to become Greater than either of them: And if not so, then it is nothing.

PROPOSITION 192

Theorem

Suppose there is a smooth curve VC (not jumping about from here to there), whose axis is VX, and with tangent VT to the vertex, and such that, taking lines from [the tangent] to the curve, parallel to the axis and equally spaced from each other, the second, fourth, sixth, eighth of them, etc. (in the even places) are as 1, 6, 30, 140, etc. (which numbers arise from the continued multiplication of these, $1 \times \frac{6}{1} \times \frac{10}{2} \times \frac{14}{3} \times \frac{18}{4}$, etc.). Then the second to the third (that is, 1 to the number that must be interposed between 1 and 6) is a semicircle to the diameter.

6.18 COMMENTS ON RIGOR [35]

Johann Kepler: “We could obtain absolute and in all respects perfect demonstrations from these books of Archimedes themselves, were we not repelled by the thorny reading thereof.”

Christiaan Huygens: “In order to achieve the confidence of the experts it is not of great interest whether we give an absolute demonstration of such a foundation of it that after having seen it they do not doubt that a perfect demonstration can be given. I am willing to concede that it should appear in a clear, elegant, and ingenious form, as in all works of Archimedes. But the first and most important thing is the mode of discovery itself, which men of learning delight in knowing. Hence it seems that we must above all follow that method by which this can be understood and presented most concisely and clearly. We then save ourselves the labor of writing, and others that of reading – those others who have no time to take notice of the enormous quantity of geometrical

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inventions which increase from day to day and in this learned century seem
to grow beyond bounds if they must use the prolix and perfect method of the
Ancients."

Bonaventura Cavalieri\textsuperscript{29} “Rigor is the affair of philosophy rather than
mathematics.”

Blaise Pascal\textsuperscript{30} “What is demonstrated by the true rules of indivisibles
could be demonstrated also with the rigor and the manner of the ancients.”

Blaise Pascal\textsuperscript{31} “We know that the truth not only by reason but also by
the heart. It is from this last source that we know the first principles and it is
in vain that reason which has no part init attempts to combat ... And it is on
our knowledge of the heart and instincts that reason necessarily rests and that
it founds on them all its discourse.”

\section*{6.19 Barrow. Method for tangents. 1666}

Barrow \cite{Barrow1666} Lecture X from the Geometrical Lectures. First given 1666
page 172–

Thus I have in some fort accomplished the chief Part of my proposed Design.
As a Supplement to which, I shall annex our Method of determining Tangents
by Calculation. Tho’ I scarcely perceive the Use of so doing, considering the
several Methods of this Nature now become common and published. I do this
at least by the Advice of a Friend\textsuperscript{32} and indeed so much the more willingly,
as it seems to be compendious and general, with respect to what else I have
handled. The Thing is thus.

Let \(AP, PM\) be right Lines given in Fig. 115. Position (whereof \(PM\) cuts
the proposed Curve in \(M\),) and let \(MT\) touch the Curve in \(M\), and cut the right
Line \(AP\) in the Point \(T\). Now to determine the length of the right Line \(PT\), I
suppose the Arch \(MN\) of the Curve to be indefinitely small, and draw the right
Lines \(NQ, NR\) parallel to \(MP, AP\); I call \(MP, m; PT, t; MR, a; NR, e;\) and
give Names to other Lines useful to our purpose, determin’d from the particular
Nature of the Curve; and then compare \(MR, NR\) expressed by Calculation in
an Equation, and by their means \(MP, PT\) themselves; observing the following
Rules at the fame Time.

1. I reject all the Terms in the Calculation, affected with any Power of \(a\) or
\(e\) or with the product of them; for these Terms will be equal to nothing.

2. After the Equation is formed, I reject all the Terms wherein are Letters
expressing constant or known Quantities; or which are not affected with \(a\), or
\(e\); for these Terms brought over to one side of the Equation will be always
equivalent to nothing.

\textsuperscript{29} Exercitationes Geometricae sex, pg 241
\textsuperscript{31} Pensees, according to page 134 of Kline.
\textsuperscript{32} Assumed to be Isaac Newton
3. I substitute \( a \) for \( m \) (\( MP \)), and \( t \) (\( PT \)) for \( e \); by which means the Quantity of \( PT \) will be found.

When any indefinitely small Particle of the Curve enters the Calculation, I substitute in its stead a Particle of the Curve properly taken; or any right Line equal to it, because of the indefinitely Smallness of the Part of the Curve.

All this will appear more evident by the following Examples.

**EXAMPLE I.**

Fig. 116 Let \( ABH \) be a right Angle, and let the Curve \( AMO \) be such, that drawing any right Line \( AK \) thro’ \( A \), cutting the right Line \( BH \) in \( K \), and the Curve \( AMO \) in \( M \), the Subtense \( AM \) may be equal to the Absciss \( BK \); it is required to draw the Tangent (at \( M \)) of this Curve, or find the Value of the right Line \( PT \).

Proceed according to the Directions above, and (drawing \( ANL \)) call \( AB, r, \) and \( AP, q \). Then is \( AG = qe \); also \( QJN = ma \). Therefore it is \( qq + ee2qe + mm + aa - 2ma = (AQq + QNq) = ANq = BLq \); that is, (rejecting according to the Rule above) \( qq2qe + mm2ma = BLq \). Again it is \( AQ : QN :: AB : BL \); that is, \( q - e : m - a :: r : BL = \frac{rm - ra}{q - e} \). Wherefore \( \frac{rm + ra - 2rmq}{qq + ee - 2qe} = BLq \). Or (casting away what is superfluous) \( \frac{rm - 2rmq}{qq - 2qe} = BLq = qq - 2qe + mm - 2ma \).

Or \( rrm - wrrma = q^4 - 2q^33 + qqmm - 2qqma - 2q^3e + 4qee - qqmm + 4qmae \), that is (rejecting as per Rule) \( -2rrma = -4q^3e - 2qqma - 2qmm + 4qmae \), or \( rrm - qqma = 2q^3e + qmm \). Or at length substituting \( m \) for \( a \) and \( t \) for \( e \), it is \( rrm - qqmm = 2q^3t - qmnt \) or \( \frac{rm - qmm}{2q^3 - qmnt} = t = PT \).

**EXAMPLE II.**

Let \( EA \) be a right Line given in Position and Magnitude, and the Curve \( EMO \) of such a Nature that drawing any how from it the right Line \( MP \) perpendicular to \( EA \), the Sum of the Cubes of \( AP \) and \( MP \) may be equal to the Cube of the right Line \( AE \).

Let \( AE = r, \) and \( AP = f \). Then is \( AQ = f + e. \) And \( ACCub. = f^3 + 3ffe + 3fee + e^3, \) or (throwing away the Superfluitues, as per direction = \( f^3 + 3ffe \). Also \( NQCub. = Cub. ma = m^33mma + 3mmaa^3 \) (that is) \( m^3 - 3mma \). Wherefore \( f^3 + 3ffe + m^33mma = (AQCub. + NQCub. = AECub. = )r^3. \) And casting off given Quantities, it is \( 3ffe3mma = 0 \) Or \( ffe = mma \), and putting \( m \) and \( f, \) for \( a \) and \( e, \) it will be \( fft = m^3, \) or \( t = \frac{m^3}{ft} \). Therefore \( PT \) is a fourth Proportional in the continu’d Ratio of \( AP \) to \( PM \).

In like manner, if it be \( APqq + MPqq = AEqq; \) you will find \( PT \) to be \( \frac{m^3}{ft}, \) or \( PM \) a fourth Proportional in the Ratio of \( AP \) to \( PM \); and so on. I do not know whether it be worth while to make this Observation of these Cycleform Lines.

**EXAMPLE III.**
Fig. 118. Let the right Line A Z be given in Position, and AX in Magnitude; also let the Curve A MO be such, that any how drawing the right Line M P perpendicular to AZ, it may be A P Cub. + PM Cub. = AX x AP x P [ref 33, 33b]

6.20 Barrow. FTC. 1666

[ref 34]
Chapter 7

Newton and Leibniz

7.1 Newton. FTC. 1669.

Of Analysis by Equations
of an infinite Number of Terms.

1. The General Method, which I had derived some considerable Time ago,
for measuring the Quantity of Curves, by Means of Series, infinite in the Num-
ber of Terms, is rather shortly explained, than accurately demonstrated in what
follows.

2. Let the Base \(AB\) of any Curve \(AD\) have \(BD\) for it’s perpendicular Ordinate; and call \(AB = x, BD = y\), and let \(a, b, c, \&c\). be given Quantities, and \(m\) and \(n\) whole Numbers. Then

The Quadrature of Simple Curves

RULE I.

3. If \(ax^n = y\); it shall be \(\frac{am}{m+n}x^{\frac{m+n}{n}} = \text{Area } ABD\).

The thing will be evident by an Example.

1. If \(x^2 = 1x^2\) \(y\), that is \(a = 1 = n\), and \(m = 2\), it shall be \(\frac{1}{3}x^3 = ABD\).
2. Suppose \(4\sqrt{x} = 4x^{\frac{1}{2}} = y\); it will be \(\frac{8}{3}x^{\frac{3}{2}} = ABD\).
3. If \(\sqrt{x^2} = x^\frac{1}{2} = y\) = it will be \(\frac{2}{3}x^3 = \frac{2}{3}\sqrt{x^3} = ABD\).
4. If \(\frac{1}{x^2} = x^{-2} = y\), that is if \(a = 1 = n\), and \(m = -2\);

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Sir Isaac Newton’s Two Treatises “Of the Quadrature of Curves” and “Analysis of Equation of an infinite Number of Terms”. By John Stewart. London, 1745. page 321–323.
CHAPTER 7. NEWTON AND LEIBNIZ

It will be \( \frac{1}{1}x^{-\frac{1}{2}} = -x^{-1}(= \frac{-1}{1}) = \alpha BD \), infinitely extended towards \( \alpha \), which the Calculation places negative, because it lies upon the other side of the line \( BD \).

5. If \( \frac{1}{\sqrt{x}} (= x^{\frac{-1}{2}}) = y \); it will be \( \frac{1}{\sqrt{x}}x^{\frac{-1}{2}} = \frac{-2}{\sqrt{x}} = BD \alpha \).

6. If \( \frac{1}{x}(= x^{-1}) = y \); it will be \( \frac{1}{x}x^{0} = \frac{1}{0} \times 1 = \infty \) = an infinite Quantity; such as is the Area of the Hyperbola upon both Sides of the Line \( BD \).

The Quadrature of Curves composed of simple ones.

RULE II.

4. If the Value of \( y \) be made up of several such Terms, the Area likewise shall be made up of the Areas which result from every one of the Terms.

The first Examples.

5. If it be \( x^2 + x^\frac{3}{2} = y \); it will be \( \frac{1}{3}x^3 + \frac{2}{5}x^\frac{5}{2} = ABD \).

For if it be always \( x^2 = BF \) and \( x^\frac{3}{2} = FD \), you will have by the preceding Rule \( \frac{1}{3}x^3 = \text{Superficie} AFB \); described by the line \( BF \); and \( \frac{2}{5}x^\frac{5}{2} = AFD \) described by \( DF \); wherefore \( \frac{1}{3}x^3 + \frac{2}{5}x^\frac{5}{2} = \text{the whole area} ABD \).

Thus if it be \( x^2 - \frac{1}{2} = y \); it will be \( \frac{1}{3}x^3 - \frac{2}{5}x^\frac{5}{2} = ABD \). And if it be \( 3x - 2x^2 + x^3 - 5x^4 = y \); it will be \( \frac{3}{2}x^2 - \frac{5}{4}x^3 + \frac{1}{4}x^4 - x^5 = ABD \).

The second Examples.

6. If \( x^{-2} + x^{-\frac{3}{2}} = y \); it will be \( -x^{-1} - 2x^{-\frac{1}{2}} = \alpha BD \). Or if it be \( x^{-2} = x^{-\frac{3}{2}} = y \); it will be \( x^{-1} + 2x^{-\frac{1}{2}} = \alpha BD \).

And if you change the Signs of the Quantities, you will have the affirmative Value \( (x^{-1} + 2x^{-\frac{1}{2}}, \text{or } x^{-1} - 2x^{-\frac{1}{2}}) \) of the Superficie \( \alpha BD \), provided the whole of it fall above the Base \( AB \).

7. But if any Part fall below (which happens when the Curve decussates or crosses its Base betwixt \( B \) and \( \alpha \) as you see here in \( \partial \)) you are to subtract that Part from the Part above the Base; and so you shall have the Value of the Difference: but if you would have their Sum, seek bot the Superficie’s separately, and add them. And the same thing I would have observed in the other Examples belonging to this Rule.
I. The Demonstration of the Quadrature of Simple Curves belonging to the
Rule the first²

Preparation for demonstrating the first Rule.

54. Let then $AD\beta$ be any Curve whose Base $AB = x$, then perpendicular
Ordinate $BD = y$, and the Area $ABD = z$, as at the Beginning. Likewise put
$B\beta = a$, $BD = v$; and the Rectangle $B\beta HK(ov)$ equal
to the Space $B\beta D$.

Therefore it is $A\beta = x + o$, and $A\beta \beta = z + ov$:
Which things being premised, assume any Relation betwixt $x$ and $z$ that you please, and seek for $y$ in the
following Manner.

Take at Pleasure $\frac{2}{3}x^3 = z$; or $\frac{4}{7} = x^3 = z^2$. Then

$x + o(A\beta)$ being substituted for $x$ and $z + ov(A\beta)$ for $z$, there arises $\frac{4}{7}$ into

$x^3 + 3xv^2 + o^3 =$ (from the Nature of the Curve) $z^2 + 2zov + o^2v^2$. And taking

away Equals ($\frac{2}{3}x^3$ and $z^2$) and dividing the Remainders by $o$, there arises $\frac{4}{7}$ into

$3x^2 + 3xo + oo = 2v + ov$. Now if we suppose $B\beta$ to be diminished infinitely and
to vanish, or $o$ to be nothing, $v$ and $y$, in that Case will be equal, and the Terms
which are multiplied by $o$ will vanish. So that there will remain $\frac{4}{7} \times 3x^2 = 2zv$,
or $\frac{2}{7}x^2 = (z + y) = \frac{2}{7}x^2y$; or $x^{\frac{1}{2}} = \frac{x^2}{y} = y$. Wherefore conversely if it be $x^{\frac{1}{2}} = y$,
it shall be $\frac{2}{7}x^2 = z$.

The Demonstration

55. Or universally, if $\frac{n}{m+n} \times ax = \frac{m+n}{m+n} = z$; or, putting $\frac{na}{m+n} = e$ and $m+n = p$,

if $cx = z$; or $c^n x^p = z^n$: Then by substituting $x + o$, for $x$, and $z + ov$ (or which
is the same $z + oy$) for $z$, there arises $c^n \int x^p + p ox^{p-1}$. &c. $= z^n + noz^{n-1}$

the other Terms, which would at length vanish being neglected. Now

taking away $c^n x^p$ and $z^n$ which are equal, and dividing the Remainders by $o$,

there remains $c^n px^{p-1} = nyz^{n-1} = \frac{nyz^{n-1}}{n^{n-1}} = \frac{nya}{ex}y$, or, by dividing by $c^n x^p$, it

shall be $px^{p-1} = \frac{nya}{ex}y$; or $pxc = nay$; or by restoring $\frac{na}{m+n}$ for $e$, and $m+n$ for

$p$, that is $m$ for $p-n$, and $na$ for $pc$, it becomes $axz = y$. Wherefore conversely, if

$axz = y$, it shall be $\frac{n}{m+n} axz^{n-1} = z$. Q.E.D.

7.2 Newton. Binomial Series. 1676.

Most worthy Sir³

Though the modesty of Mr Leibniz, in the extracts from his letter which you
have lately sent me, pays great tribute to our countrymen for a certain theory

²We have skipped to page 340 of Stewart.
³meaning, multiplied by
of infinite series, about which there now begins to be some talk, yet I have no
doubt that he has discovered not only a method for reducing any quantities
whatever to such series, as he asserts, but also various shortened forms, perhaps
like our own, if not even better. Since, however, he very much wants to know
what has been discovered in this subject by the English, and since I myself fell
upon this theory some years ago, I have sent you some of those things which
occurred to me in order to satisfy his wishes, at any rate in part.

Fractions are reduced to infinite series by division; and radical quantities by
extraction of the roots, by carrying out those operations in the symbols just as
they are commonly carried out in decimal numbers. These are the foundations of
these reductions: but extractions of roots are much shortened by this theorem,
\[(P + PQ)^{m/n} = P^{m/n} + \frac{m-n}{2n} BQ + \frac{m-2n}{3n} CQ + \frac{m-3n}{4n} DQ + \text{etc.,}\]
where \(P + PQ\) signifies the quantity whose root or even any power, or the root
of a power, is to be found; \(P\) signifies the first term of that quantity, \(Q\) the
remaining terms divided by the first, and \(m/n\) the numerical index of the power
of \(P + PQ\), whether that power is integral or (so to speak) fractional, whether
positive or negative. For as analysts, instead of \(aa\), \(aaa\), etc., are accustomed
to write \(a^2\), \(a^3\), etc.; I write \(a^{1/2}\), \(a^{3/2}\), \(a^{5/3}\), and
instead of \(1/a\), \(1/aa\), \(1/a^3\), I write \(a^{-1}\), \(a^{-2}\), \(a^{-3}\). And so for
\[
\sqrt[3]{c : \{(a^3 + bbx)/(a^3 + bbx)\}}
\]
I write \(a^{1/3}b(a^3 + bbx)^{-2/3}\): in which last case, if \((a^3 + bbx)^{-2/3}\) is supposed to be
\((P + PQ)^{m/n}\) in the Rule, then \(P\) will be equal to \(a^3\), \(Q\) to \(bbx/a^3\), \(m\) to
\(-2\), and \(n\) to 3. Finally, for the terms found in the quotient in the course of
the working I employ \(A\), \(B\), \(C\), \(D\), etc., namely, \(A\) for the first term, \(P^{m/n}\); \(B\)
for the second term, \((m/n)AQ\); and so on. For the rest, the use of the rule will
appear from the examples.

Example 1.
\[
\sqrt[(c^2 + x^2)]{(c^2 + x^2)} = c + \frac{x^2}{2c} - \frac{x^4}{8c^3} + \frac{x^6}{16c^5} - \frac{5x^8}{128c^7} + \frac{7x^{10}}{256c^9} + \text{etc.}
\]
For in this case \(P = c^2\), \(Q = x^2/c^2\), \(m = 1\), \(n = 2\), \(A (= P^{m/n} = (cc)^{1/2}) = c\),
\(B (= (m/n)AQ) = x^2/2c\), \(C (= \frac{m-n}{2n}BQ) = -\frac{x^4}{8c^3}\); and so on.

...
I can hardly tell with what pleasure I have read the letters of those very distinguished men Leibniz and Tschirnhaus. Leibniz’s method for obtaining convergent series is certainly very elegant, and it would have sufficiently revealed the genius of its author, even if he had written nothing else. But what he has scattered elsewhere throughout his letter is most worthy of his reputation – it leads us also to hope for very great things from him. The variety of ways by which the same goal is approached has given me the greater pleasure, because three methods of arriving at series of that kind had already become known to me, so that I could scarcely expect a new one to be communicated to us.

One of mine I have described before; I now add another, namely, that by which I first chanced on these series – for I chanced on them before I knew the divisions and extractions of roots which I now use. And an explanation of this will serve to lay bare, what Leibniz desires from me, the basis of the theorem set forth near the beginning of the former letter.

At the beginning of my mathematical studies, when I had met with the works of our celebrated Wallis, on considering the series by the intercalation of which he himself exhibits the area of the circle and the hyperbola, the fact that, in the series of curves whose common base or axis is $x$, and the ordinate $(1 - x^2)^{\frac{1}{2}}, (1 - x^2)^{\frac{1}{3}}, (1 - x^2)^{\frac{1}{4}}, (1 - x^2)^{\frac{1}{5}}, (1 - x^2)^{\frac{1}{6}}, (1 - x^2)^{\frac{1}{7}},$ etc.,

if the areas of every other of them, namely

$x, x - \frac{1}{3}x^3, x - \frac{2}{3}x^3 + \frac{1}{5}x^5, x - \frac{2}{3}x^3 - \frac{3}{5}x^5 - \frac{1}{7}x^7,$ etc.

could be interpolated, we should have the areas of the intermediate ones, of which the first $(1 - x^2)^{\frac{1}{2}}$ is the circle: in order to interpolate these series I noted that in all of them the first term was $x$ and that the second terms $\frac{2}{3}x^3, \frac{1}{5}x^5, \frac{3}{5}x^5,$ etc., were in arithmetical progression, and hence that the first two terms of the series to be intercalated ought to be $x - \frac{1}{4}(\frac{1}{2}x^3), x - \frac{1}{4}(\frac{3}{5}x^5),$ etc. To intercalated the rest I began to reflect that the denominators $1, 3, 5, 7,$ etc. were in arithmetical progression, so that the numerical coefficients of the numerators only were still in need of investigation. But in the alternately given areas these were the figures of the power of the number 11, namely of these, $11^0, 11^1, 11^2, 11^3, 11^4,$ that is, first 1; then 1, 1, thirdly 1, 2, 1; fourthly 1, 3, 3, 1; fifthly 1, 4, 6, 4, 1, etc. And so I began to inquire how the remaining figures in these series could be derived from the first two given figures, and I found that on putting $m$ for the second figure, the rest would be produced by continual multiplication of the terms of this series,

$$\frac{m - 0}{1} \times \frac{m - 1}{2} \times \frac{m - 2}{3} \times \frac{m - 3}{4} \times \frac{m - 4}{5},$$

For example, let $m = 4$, and $4 \times \frac{1}{4}(m - 1)$, that is 6 will be the third term, and $6 \times \frac{1}{4}(m - 2)$, that is 4 the fourth, and $4 \times \frac{1}{4}(m - 3)$, that is 1 the fifth, and $1 \times \frac{1}{4}(m - 4)$, that is 0 the sixth, at which term in this case the series stops. Accordingly, I applied this rule for interposing series among series, and since,
for the circle, the second terms was \( \frac{1}{7} (\frac{1}{2} x^3) \), I put \( m = \frac{1}{2} \), and the terms arising were

\[
\frac{1}{2} \times \frac{\frac{1}{2} - 1}{2} \text{ or } - \frac{1}{8} \times \frac{\frac{1}{2} - 2}{3} \text{ or } + \frac{1}{16} \times \frac{\frac{1}{2} - 3}{4} \text{ or } - \frac{5}{128},
\]

and so to infinity. Whence I came to understand that the area of the circular segment which I wanted was

\[
x - \frac{\frac{1}{2} x^3}{3} - \frac{\frac{1}{2} x^5}{5} - \frac{\frac{1}{16} x^7}{7} - \frac{\frac{5}{128} x^9}{9} \text{ etc.}
\]

And by the same reasoning the areas of the remaining curves, which were to be inserted, were likewise obtained: as also the area of the hyperbola and the other alternate curves in this series \((1 + x^2)^{\frac{1}{2}}, (1 + x^2)^{\frac{3}{2}}, (1 + x^2)^{\frac{5}{2}}, (1 + x^2)^{\frac{7}{2}}, \text{ etc.} \) And the same theory serves to intercalate other series, and that through intervals of two or more terms when they are absent at the same time. This was my first entry upon these studies, and it had certainly escaped my memory, had I not a few weeks ago cast my eye back on some notes.

But when I had learnt this, I immediately began to consider that the terms

\[(1 - x^2)^{\frac{2}{3}}, (1 - x^2)^{\frac{4}{3}}, (1 - x^2)^{\frac{6}{3}}, (1 - x^2)^{\frac{8}{3}}, \text{ etc.,}\]

that is to say,

\[1, 1 - x^2, x - 2x^2 + x^4, 1 - 3x^2 + 3x^4 - x^6, \text{ etc.}\]

could be interpolated in the same way as the areas generated by them: and that nothing else was required for this purpose but to omit the denominators \( l, 3, 5, 7, \text{ etc.} \), which are in the terms expressing the areas; this means that the coefficients of the terms of the quantity to be intercalated \((1-x^2)^{\frac{1}{2}}, \text{ or } (1-x^2)^{\frac{3}{2}}, \text{ or in general } (1-x^2)^m\), arise by the continued multiplication of the terms of this series

\[m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}, \text{ etc.}\]

so that (for example)

\[(1 - x^2)^{\frac{1}{2}} \text{ was the value of } 1 - \frac{1}{2} x^2 - \frac{1}{8} x^4 - \frac{1}{16} x^6, \text{ etc.,}\]

\[(1 - x^2)^{\frac{3}{2}} \text{ of } 1 - \frac{3}{2} x^2 + \frac{3}{8} x^4 + \frac{1}{16} x^6, \text{ etc.,}\]

and

\[(1 - x^2)^{\frac{5}{2}} \text{ of } 1 - \frac{5}{3} x^2 - \frac{5}{9} x^4 - \frac{5}{81} x^6, \text{ etc.,}\]

So then the general reduction of radicals into infinite series by that rule, which I laid down at the beginning of my earlier letter) became known to me, and that before I was acquainted with the extraction of roots. But once this was known,
that other could not long remain hidden from me. For in order to test these processes, I multiplied
\[
1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6, \text{ etc.,}
\]
into itself; and it became \(1 - x^2\), the remaining terms vanishing by the continuation of the series to infinity. And even so \(1 - \frac{1}{4}x^2 - \frac{1}{8}x^4 - \frac{5}{32}x^6\), etc., multiplied twice into itself also produced \(1 - x^2\). And as this was not only sure proof of these conclusions so too it guided me to try whether, conversely, these series, which it thus affirmed to be roots of the quantity \(1 - x^2\), might not be extracted out of it in an arithmetical manner. And the matter turned out well...

7.3 Gottlieb Leibniz, Derivatives. 1684.

A NEW METHOD FOR MAXIMA AND MINIMA AS WELL AS TANGENTS, WHICH IS NEITHER IMPEDED BY FRACTIONAL NOR IRRATIONAL QUANTITIES, AND A REMARKABLE TYPE OF CALCULUS FOR THEM.

BY G.W.L.

Let an axis \(AX\) and several curves such as \(VV, WW, YY, ZZ\) be given of which the ordinates \(VX, WX, YX, ZX, \) perpendicular to the axis, are called \(v, w, y, z\) respectively. The segment \(AX\), cut off from the axis \([\text{abscissa ab axe}]\) is called \(x\). Let the tangents be \(VB, WC, YD, ZE\), intersecting the axis respectively at \(B, C, D, E\). Now some straight line selected arbitrarily is called \(dx\), and the line which is to \(dx\) as \(v\) (or \(w\), or \(y\), or \(z\)) is to \(XB\) (or \(XC\), or \(XD\), or \(XE\)) is called \(dv\) (or \(dw\), or \(dy\), or \(dz\)), or the difference of these \(v\) (or \(w\), or \(y\), or \(z\)). Under these assumptions we have the following rules of the calculus.

If \(a\) is a given constant, then \(da = 0\), and \(d(ax) = adx\). If \(y = v\) (that is, if the ordinate of any curve \(YY\) is equal to any corresponding ordinate of the curve \(VV\), then \(dy = dv\).

Now addition and subtraction: if \(z - y + w + z = v\), then \(d(z - y + w + x) = dv = dz - dy + dw + dx\). Multiplication: \(d(xv) = xdv + vdx\), or, setting \(y = xv\), \(dy = xdv + vzdx\). It is indifferent whether we take a formula such as \(xv\) or its replacing letter such as \(y\). It is to be noted that \(x\) and \(dx\) are treated in this calculus in the same way as \(y\) and \(dy\), or any other indeterminate letter with

\[6\] [ref 38, 38b], \textit{Nova methodus pro maximis et minimis, ...} 1684, Acta eruditorum, 467-72

Struik translation, pages 272–279. Item [38]

\[7\] Figure adapted from Struik.
its difference. It is also to be noted that we cannot always move backward from
a differential equation without some caution, something which we shall discuss
elsewhere.

Now division: \( \frac{dz}{y} \) or (if \( z = \frac{v}{y} \)) \( dz = \frac{\pm v dy + y dv}{yy} \)

The following should be kept well in mind about the signs. When in the
calculus for a letter simply its differential is substituted, then the signs are pre-
served; for \( z \) we write \( dz \), for \(-z\) we write \(-dz\), as appears from the previously
given rule for addition and subtraction. However, when it comes to an expla-
nation of the values, that is, when the relation of \( z \) to \( x \) is considered, then
we can decide whether \( dz \) is a positive quantity or less than zero. When the
latter occurs, then the tangent \( ZE \) is not directed toward \( A \), but in the op-
posite direction, down from \( X \). This happens when the ordinates \( z \) decrease
with increasing \( x \). And since the ordinates \( v \) sometimes increase and sometimes
decrease, \( dv \) will sometimes be positive and sometimes be negative; in the first
case the tangent \( VB \) is directed toward \( A \), in the latter it is directed in the
opposite sense. None of these cases happens in the intermediate position at \( M \),
at the moment when \( v \) neither increases nor decreases, but is stationary. Then
\( dv = 0 \), and it does not matter whether the quantity is positive or negative,
since \( +0 = -0 \). At this place \( v \), that is, the ordinate \( LM \), is maximum (or,
when the convexity is turned to the axis, minimum), and the tangent to the
curve at \( M \) is directed neither in the direction from \( X \) up to \( A \), to approach the
axis, nor down to the other side, but is parallel to the axis. When \( dv \) is infinite
with respect to \( dx \), then the tangent is perpendicular to the axis, that is, it is
the ordinate itself. When \( dv = dx \), then the tangent makes half a right angle
with the axis. When with increasing ordinates \( v \) its increments or differences \( dv \)
also increase (that is, when \( dv \) is positive, \( ddv \), the difference of the differences,
is also positive, and when \( dv \) is negative, \( ddv \) is also negative), then the curve
turns toward the axis its concavity, in the other case its convexity. Where the
increment is maximum or minimum, or where the increments from decreasing
turn into increasing, or the opposite, there is a point of inflection. Here con-
cavity and convexity are interchanged, provided the ordinates too do not turn
from increasing into decreasing or the opposite, because then the concavity or
convexity would remain. However, it is impossible that the increments continue
to increase or decrease, but the ordinates turn from increasing into decreasing,
or the opposite. Hence a point of inflection occurs when \( ddv = 0 \) while neither
\( v \) nor \( dv = 0 \). The problem of finding inflection therefore has not, like that of
finding a maximum, two equal roots, but three. This all depends on the correct
use of the signs.

Sometimes it is better to use ambiguous signs, as we have done with the
division, before it is determined what the precise sign is. When with increasing
\( xv/y \) increases (or decreases), then the ambiguous signs in \( \frac{dv}{y} = \frac{\pm v dy + y dv}{yy} \)
must be determined in such a way that this fraction is a positive (or negative) quantity.
But \( \mp \) means the opposite of \( \pm \), so that when one is \( + \) the other is \( - \). There
also may be several ambiguities in the same computation, which I distinguish
7.3. GOTTLEIB LEIBNIZ, DERIVATIVES. 1684.

by parentheses. For example, let \( \frac{x}{y} + \frac{y}{z} + \frac{z}{v} = w \); then we must write

\[
\pm vdy \pm ydv/yy + (\pm)ydz(\mp)zdy/zz + ((\pm))xdv((\mp))vdx/vv = dv,
\]

so that the ambiguities in the different terms may not be confused. We must take notice that an ambiguous sign with itself gives +, with its opposite gives −, while with another ambiguous sign it forms a new ambiguity depending on both.

Powers:

\[
dx^a = ax^a - 1dx; \text{ for example, } dx^3 = 3x^2dx. \quad d\frac{1}{x^a} = -\frac{ax}{x^{a+1}};
\]

for example, if \( w = \frac{1}{x} \) then \( dw = -\frac{dx}{x^2} \).

Roots:

\[
d\sqrt[\sqrt{b}]{x^a} = \frac{a}{\sqrt{b}}dx\sqrt[\sqrt{b}]{x^{a-b}} \quad (\text{hence } d\frac{1}{\sqrt{y}} = \frac{dy}{2\sqrt{y}}, \text{ for in this case } a = 1, b = 2), \text{ therefore } \frac{a}{\sqrt{b}}\sqrt[\sqrt{b}]{x^a-b} = \frac{1}{\sqrt{y}} \frac{dy}{2\sqrt{y}} \text{, but } y^{-1} \text{ is the same as } \frac{1}{y}; \text{ from the nature of exponents in a geometric progression, and } \frac{1}{\sqrt{y}} = \frac{1}{\sqrt{y}} \text{ is the same as } \frac{1}{\sqrt{y}}; \text{ due to the nature of exponents in a geometric progression, and } \frac{1}{\sqrt{y}} = \frac{1}{\sqrt{y}} \text{ is the same as } \frac{1}{\sqrt{y}}. \text{ The law for integral powers would have been sufficient to cover the case of fractions as well as roots, for a power becomes a fraction when the exponent is negative, and changes into a root when the exponent is fractional. However, I prefer to draw these conclusions myself rather than relegate their deduction to others, since they are quite general and occur often. In a matter that is already complicated in itself it is preferable to facilitate the operations.}

Knowing thus the Algorithm (as I may say) of this calculus, which I call differential calculus, all other differential equations can be solved by a common method. We can find maxima and minima as well as tangents without the necessity of removing fractions, irrationals, and other restrictions, as had to be done according to the methods that have been published hitherto. The demonstration of all this will be easy to one who is experienced in these matters and who considers the fact, until now not sufficiently explored, that \( dx, dy, dv, dw \), \( dz \) can be taken proportional to the momentary differences, that is, increments or decrements, of the corresponding \( x, y, v, w, z \). To any given equation we can thus write its differential equation. This can be done by simply substituting for each term (that is, any part which through addition or subtraction contributes to the equation) its differential quantity. For any other quantity (not itself a term, but contributing to the formation of the term) we use its differential quantity, to form the differential quantity of the term itself, not by simple substitution, but according to the prescribed Algorithm. The methods published before have no such transition. They mostly use a line such as \( DX \) or of similar kind, but not the line \( dy \) which is the fourth proportional to \( DX, DY, dx \)—something quite confusing. From there they go on removing fractions and irrationals (in which undetermined quantities occur). It is clear that our method also covers transcendental curve’s—those that cannot be reduced by algebraic computation, or have no particular degree and thus holds in a most general way without any particular and not always satisfied assumptions.

We have only to keep in mind that to find a tangent means to draw a line that connects two points of the curve at an infinitely small distance, or the continued
side of a polygon with an infinite number of angles, which for us takes the place of the curve. This infinitely small distance can always be expressed by a known differential like \( dv \), or by a relation to it, that is, by some known tangent. In particular, if \( y \) were a transcendental quantity, for instance the ordinate of a cycloid, and it entered into a computation in which \( z \), the ordinate of another curve, were determined, and if we desired to know \( dz \) or by means of \( dz \) the tangent of this latter curve, then we should by all means determine \( dz \) by means of \( dy \), since we have the tangent of the cycloid. The tangent to the cycloid itself, if we assume that we do not yet have it, could be found in a similar way from the given property of the tangent to the circle.

Now I shall propose an example of the calculus, in which I shall indicate division by \( x \): \( y \), which means the same as \( x \) divided by \( y \), or \( x/y \).

Let two points \( C \) and \( E \) be given and a line \( SS \) in the same plane. It is required to find a point \( F \) on \( SS \) such that when \( E \) and \( C \) are connected with \( F \) the sum of the rectangle \( CF \) and a given line \( h \) and the rectangle of \( FE \) and a given line \( r \) are as small as possible. In other words, if \( SS \) is a line separating two media, and \( h \) represents the density of the medium on one side of \( C \) (say water), \( r \) that of the medium on the side of \( E \) (say air), then we ask for the point \( F \) such that the path from \( C \) to \( E \) via \( F \) is the shortest possible. Let us assume that all such possible sums of rectangles, or all possible paths, are represented by the ordinates \( KV \) of curve \( VV \) perpendicular to the line \( GK \). We shall call these ordinates \( w \). Then it is required to find their minimum \( NM \). Since \( C \) and \( E \) are given, their perpendiculars to \( SS \) are also given, namely \( CP \) (which we call \( c \)) and \( EQ \) (which we call \( e \)); moreover \( PQ \) (which we call \( p \)) is given. We denote \( QF = GN \) (or \( AX \)) by \( x \), \( CF \) by \( f \), and \( EF \) by \( g \). Then \( FP = p - x \), \( f = \sqrt{cc + pp - 2px + xx} \) or \( = \sqrt{l} \) for short; \( g = \sqrt{ee + xx} \) or \( = \sqrt{m} \) for short.

Hence

\[ w = h\sqrt{l} + r\sqrt{m}. \]

The differential equation (since \( dw = 0 \) in the case of a minimum) is, according to our calculus,

\[ 0 = +h \frac{dl}{l} : 2\sqrt{l} + r \frac{dm}{m} : 2\sqrt{m}. \]

But \( dl = -2(p-x) \) \( dx \), \( dm = 2d \) \( dx \); hence

\[ h(p-x) : f = fx : g. \]

When we now apply this to dioptrics, and take \( f \) and \( g \), that is, \( CF \) and \( EF \), equal to each other (since this refraction at the point \( F \) is the same no matter how long the line \( CF \) may be), then \( h(p-x) = rx \) or \( h : r = x : (p-x) \), or \( h : r = QF : FP \); hence the sines of the angles of incidence, and of refraction, \( FP \) and \( QF \), are in inverse ratio to \( r \) and \( h \), the densities of the media in which the incidence and the refraction take place. However, this density is not to be
understood with respect to use, but to the resistance which the light rays meet. Thus we have a demonstration of the computation exhibited elsewhere in these Acta [1682], where we presented a general foundation of optics, catoptrics, and dioptrics. Other very learned men have sought in many devious ways what someone versed in this calculus can accomplish in these lines as if by magic. ...

7.4 Gottleib Leibniz, On his Discovery of Differential Calculus

When my infinitesimal calculus, which includes the calculus of differences and sums, had appeared and spread, certain over-precise veterans began to make trouble; just as once long ago the Sceptics opposed the Dogmatics, as is seen from the work of Empiricus against the mathematicians (i.e., the dogmatics), and such as Francisco Sanchez, the author of the book Quod nihil scitur, brought against Clavius; and his opponents to Cavalieri, and Thomas Hobbes to all geometers, and just lately such objections as are made against the quadrature of the parabola by Archimedes by that renowned man, Dethlevus Cluver. When then our method of infinitesimals, which had become known by the name of the calculus of differences, began to be spread abroad by several examples of its use, both of my own and also of the famous brothers Bernoulli, and more especially by the elegant writings of that illustrious Frenchman, the Marquis d’Hopital, just lately a certain erudite mathematician, writing under an assumed name in the scientific Journal de Trevoux, appeared to find fault with this method. But to mention one of them by name, even before this there arose against me in Holland Bernard Nieuwentiit, one indeed really well equipped both in learning and ability, but one who wished rather to become known by revising our methods to some extent than by advancing them. Since I introduced not only the first differences, but also the second, third and other higher differences, inassignable or incomparable with these first differences, he wished to appear satisfied with the first only; not considering that the same difficulties existed in the first as in the others that followed, nor that wherever they might be overcome in the first, they also ceased to appear in the rest. Not to mention how a very learned young man, Hermann of Basel, showed that the second and higher differences were avoided by the former in name only, and not in reality; moreover, in demonstrating theorems by the legitimate use of the first differences, by adhering to which he might have accomplished some useful work on his own account, he fails to do so, being driven to fall back on assumptions that are admitted by no one; such as that something different is obtained by multiplying 2 by \( m \) and by multiplying \( m \) by 2; that the latter was impossible in any case in which the former was possible; also that the square or cube of a quantity is not a quantity or Zero.

In it, however, there is something that is worthy of all praise, in that he desires that the differential calculus should be strengthened with demonstrations, so that it may satisfy the rigorists; and this work he would have procured from me already, and more willingly, if, from the fault-finding everywhere interspersed, the wish had not appeared foreign to the manner of those who desire the truth rather than fame and a name.

It has been proposed to me several times to confirm the essentials of our calculus by demonstrations, and here I have indicated below its fundamental principles, with the intent that any one who has the leisure may complete the work. Yet I have not seen up to the present any one who would do it. For what the learned Hermann has begun in his writings, published in my defence against Nieuwentiit, is not yet complete.

For I have, beside the mathematical infinitesimal calculus, a method also for use in Physics, of which an example was given in the *Nouvelles de la Republique des Lettres*; and both of these I include under the Law of Continuity; and adhering to this, I have shown that the rules of the renowned philosophers Descartes and Malebranche were sufficient in themselves to attack all problems on Motion.

I take for granted the following postulate:

*In any supposed transition, ending in any terminus, it is permissible to institute a general reasoning, in which the final terminus may also be included.*

For example, if A and B are any two quantities, of which the former is the greater and the latter is the less, and while B remains the same, it is supposed that A is continually diminished, until A becomes equal to B; then it will be permissible to include under a general reasoning the prior cases in which A was greater than B, and also the ultimate case in which the difference vanishes and A is equal to B. Similarly, if two bodies are in motion at the same time, and it is assumed that while the motion of B remains the same, the velocity of A is continually diminished until it vanishes altogether, or the speed of A becomes zero; it will be permissible to include this case with the case of the motion of B under one general reasoning. We do the same thing in geometry, when two straight lines are taken, produced in any manner, one VA being given in position or remaining in the same site, the other BP passing through a given point P, and varying in position while the point P remains fixed; at first indeed converging toward the line VA and meeting it in the point C; then, as the angle of inclination VGA is continually diminished, meeting VA in some more remote point (C), until at length from BP, through the position (B)P, it comes βP, in which the straight line no longer converges toward VA, but is parallel to it, and C is an impossible or imaginary point. With this supposition it is permissible to include under some one general reasoning not only all the intermediate cases such as (B)P but also the ultimate case βP.
Hence also it comes to pass that we include as one case ellipses and the parabola, just as if A is considered to be one focus of an ellipse (of which V is the given vertex), and this focus remains fixed, while the other focus is variable as we pass from ellipse to ellipse, until at length (in the case when the line BP, by its intersection with the line VA, gives the variable focus) the focus C becomes evanescent or impossible, in which case the ellipse passes into a parabola. Hence it is permissible with our postulate that a parabola should be considered with ellipses under a common reasoning. Just as it is common practice to make use of this method in geometrical constructions, when they include under one general construction many different cases, noting that in a certain case the converging straight line passes into a parallel straight line, the angle between it and another straight line vanishing.

Moreover, from this postulate arise certain expressions which are generally used for the sake of convenience, but seem to contain an absurdity, although it is one that causes no hindrance, when its proper meaning is substituted. For instance, we speak of an imaginary point of intersection as if it were a real point, in the same manner as in algebra imaginary roots are considered as accepted numbers. Hence, preserving the analogy, we say that, when the straight line BP ultimately becomes parallel to the straight line VA, even then it converges toward it or makes an angle with it, only that the angle is then infinitely small; similarly, when a body ultimately comes to rest, it is still said to have a velocity, but one that is infinitely small; and, when one straight line is equal to another, it is said to be unequal to it, but that the difference is infinitely small; and that a parabola is the ultimate form of an ellipse, in which the second focus is at an infinite distance from the given focus nearest to the given vertex, or in which the ratio of PA to AC, or the angle BCA, is infinitely small.

Of course it is really true that things which are absolutely equal have a difference which is absolutely nothing; and that straight lines which are parallel never meet, since the distance between them is everywhere the same exactly; that a parabola is not an ellipse at all, and so on. Yet, a state of transition may be imagined, or one of evanescence, in which indeed there has not yet arisen exact equality or rest or parallelism, but in which it is passing into such a state, that the difference is less than any assignable quantity; also that in this state there will still remain some difference, some velocity, some angle, but in each case one that is infinitely small; and the distance of the point of intersection, or the variable focus, from the fixed focus will be infinitely great, and the parabola may be included under the heading of an ellipse (and also in the same manner and by the same reasoning under the heading of a hyperbola), seeing that those things that are found to be true about a parabola of this kind are in no way different, for any construction, from those which can be stated by treating the parabola rigorously.

Truly it is very likely that Archimedes, and one who seems so have surpassed him, Conon, found out their wonderfully elegant theorems by the help of such ideas; these theorems they completed with reductio ad absurdum proofs,

\footnote{in the sense of “vanishing into the far distance”}
by which they at the same time provided rigorous demonstrations and also concealed their methods. Descartes very appropriately remarked in one of his writings that Archimedes used as it were a kind of metaphysical reasoning (Caramuel would call it metageometry), the method being scarcely used by any of the ancients (except those who dealt with quadratrices); in our time Cavalieri has revived the method of Archimedes, and afforded an opportunity for others to advance still further. Indeed Descartes himself did so, since at one time he imagined a circle to be a regular polygon with an infinite number of sides, and used the same idea in treating the cycloid; and Huygens too, in his work on the pendulum, since he was accustomed to confirm his theorems by rigorous demonstrations; yet at other times, in order to avoid too great proximity, he made use of infinitesimals; as also quite lately did the renowned La Hire.

For the present, whether such a state of instantaneous transition from inequality to equality, from motion to rest, from convergence to parallelism, or anything of the sort, can be sustained in a rigorous or metaphysical sense, or whether infinite extensions successively greater and greater, or infinitely small ones successively less and less, are legitimate considerations, is a matter that I own to be possibly open to question; but for him who would discuss these matters, it is not necessary to fall back upon metaphysical controversies, such as the composition of the continuum, or to make geometrical matters depend thereon. Of course, there is no doubt that a line may be considered to be unlimited in any manner, and that, if it is unlimited on one side only, there can be added to it something that is limited on both sides. But whether a straight line of this kind is to be considered as one whole that can be referred to computation, or whether it can be allocated among quantities which may be used in reckoning, is quite another question that need not be discussed at this point.

It will be sufficient if, when we speak of infinitely great (or more strictly unlimited), or of infinitely small quantities (i.e., the very least of those within our knowledge), it is understood that we mean quantities that are indefinitely great or indefinitely small, i.e., as great as you please, or as small as you please, so that the error that any one may assign may be less than a certain assigned quantity. Also, since in general it will appear that, when any small error is assigned, it can be shown that it should be less, it follows that the error is absolutely nothing: an almost exactly similar kind of argument is used in different places by Euclid, Theodosius and others; and this seemed to them to be a wonderful thing, although it could not be denied that it was perfectly true that, from the very thing that was assumed as an error, it could be inferred that the error was non-existent. Thus, by infinitely great and infinitely small, we understand something indefinitely great, or something indefinitely small, so that each conducts itself as a sort of class, and not merely as the last thing of a class. If any one wishes to understand these as the ultimate things, or as truly infinite, it can be done, and that too without falling back upon a controversy about the reality of extensions, or of infinite continuums in general, or of the infinitely small, ay, even though he think that such things are utterly impossible; it will be sufficient simply to make use of them as a tool that has advantages for
the purpose of the calculation, just as the algebraists retain imaginary roots with
great profit. For they contain a handy means of reckoning, as can manifestly be
verified in every case in a rigorous manner by the method already stated.

But it seems right to show this a little more clearly, in order that it may
be confirmed that the algorithm, as it is called, of our differential calculus, set
forth by me in the year 1684, is quite reasonable. First of all, the sense in which
the phrase “$dy$ is the element of $y$,” is to be taken will best be understood by
considering a line $AY$ referred to a straight line $AX$ as axis.

Let the curve $AY$ be a parabola, and let the tangent at the vertex $A$ be taken
as the axis. If $AX$ is called $x$, and $AY$, $y$, and the
latus-rectum is $a$, the equation to the parabola will
be $xx - ay$, and this holds good at every point. Now,
let $A_1X = x$, and $1X_1Y = y$ and from the point $1Y$
let fall a perpendicular $1T$ to some greater ordinate
$2X_2Y$ that follows, and let $1Y_2X$, the difference
between $A_1X$ and $A_2X$ be called $dx$; and similarly,
let $D_2Y$, the difference between $1X_1Y$ and $2X_2Y$,
be called $dy$.

Then, since $y - xx : a$, by the same law, we have

$$y + dy = xx + 2x\, dx + dx\, dx, : a;$$

and taking away the $y$ from the one side and the $xx : a$ from the other, we have left

$$dy : dx = 2x + dx : a;$$

and this is a general rule, expressing the ratio of the difference of the ordinates
to the difference of the abscissae, or, if the chord $1Y_2Y$ is produced until it
meets the axis in $T$, then the ratio of the ordinate $1_1Y$ to $T_1X$, the part of
the axis intercepted between the point of intersection and the ordinate, will be
as $2x + dx$ to $a$. Now, since by our postulate it is permissible to include under
the one general reasoning the case also in which the ordinate $2X_2Y$ is moved
up nearer and nearer to the fixed ordinate $1X_1Y$ until it ultimately coincides
with it, it is evident that in this case $dx$ becomes equal to zero and should be
neglected, and thus it is clear that, since in this case $T_1Y$ is the tangent, $1X_1Y$
is to $T_1X$ as $2x$ is to $a$.

Hence, it may be seen that there is no need in the whole of our differential
calculus to say that those things are equal which have a difference that is
ininitely small, but that those things can be taken as equal that have not any
difference at all, provided that the calculation is supposed to be general, in-
cluding both the cases in which there is a difference and in which the difference
is zero; and provided that the difference is not assumed to be zero until the
calculation is purged as far as is possible by legitimate omissions, and reduced
to ratios of non-evanescent quantities, and we finally come to the point where
we apply our result to the ultimate case.
Similarly, if \( x^3 = aay \), then we have
\[
x^3 + 3xx\, dx + 3x\, dx\, dx + dx\, dx\, dx = aay + aa\, dy
\]
or cancelling from each side,
\[
3xx\, dx + 3x\, dx\, dx + dx\, dx\, dx = aa\, dy
\]
or
\[
3xx + 3x\, dx + dx\, dx : aa = dy : dx = X_1Y : T_1X;
\]
hence, when the difference vanishes, we have
\[
3xx : aa = X_1Y : T_1X.
\]

But if it is desired to retain \( dy \) and \( dx \) in the calculation, so that they may represent non-evanescent quantities even in the ultimate case, let any assignable straight line be taken as \( (dx) \), and let the straight line which bears to \( (dx) \) the ratio of \( y \) or \( X_1Y \) to \( X_1XT \) be called \( (dy) \); in this way \( dy \) and \( dx \) will always be assignables bearing to one another the ratio of \( D_2Y \) to \( D_1Y \), which latter vanish in the ultimate case.

[Leibniz here gives a correction for a passage in the Acta Eruditorum, which is unintelligible without the context.]

On these suppositions, all the rules of our algorithm, as set out in the Acta Eruditorum for October 1684, can be proved without much trouble.

Let the curves \( YY, \, VV, \, ZZ \) be referred to the same axis \( AXX \); and to the abscissae \( A_1X (= x) \) and \( A_2X (= x+dx) \) let there correspond the ordinates \( 1X_1y (= y) \) and \( 2X_2y (= y+dy) \), and also the ordinates \( 1X_1V (= v) \) and \( 2X_2V (= v+dv) \), and the ordinates \( 1X_1Z (= z) \) and \( 2X_2Z (= z+dz) \). Let the chords \( 1Y_2Y, \, 1V_2V, \, 1Z_2Z \), when produced meet the axis \( AXX \) in \( T, U, W \). Take any straight line you will as \( (dx) \), and, while the point \( 1X \) remains fixed and the point \( 2X \) approaches \( 1X \) in any manner, let this remain constant, and let \( (dy) \) be another line which bears to \( (dx) \) the ratio of \( y \) to \( 1X \, T \), or of \( dy \) to \( dx \); and similarly, let \( (dv) \) be to \( (dx) \) as \( v \) to \( 1X \, U \) or \( dv \) to \( dx \); also let \( (dz) \) be to \( (dx) \) as \( z \) to \( 1X \, W \) or \( dz \) to \( dx \); then \( (dx) \), \( (dy) \), \( (dz) \), \( (dv) \) will always be ordinary or assignable straight lines.

Nor for Addition and Subtraction we have the following:

If \( y - z = v \), then \( (dy) - (dz) = (dv) \).
This I prove thus: $y + dy - z - dz = v + dv$, (if we suppose that as $y$ increases, $z$ and $v$ also increase; otherwise for decreasing quantities, for $z$ say, $-dz$ should be taken instead of $dz$, as I mentioned once before): hence, rejecting the equals, namely $y - z$ from one side, and $v$ from the other, we have $dy - dz = dv$, and therefore also $dy - dz : dx = dv : dx$. But $dy : dx, dz : dx, dv : dx$ are respectively equal to $(d)y : (d)x, (d)z : (d)x$, and $(d)v : (d)x$. Similarly, $(d)z : (d)y$ and $(d)v : (d)y$ are respectively equal to $dz : dy$ and $dv : dy$. Hence, $(d)y - (d)z, (d)x = (d)v : (d)x$; and thus $(d)y - (d)z$ is equal to $(d)v$, which was to be proved; or we may write the result as $(d)v : (d)y = l - (d)z : (d)y$.

This rule for addition and subtraction also comes out by the use of our postulate of a common calculation, when $X$ coincides with $Y$, and $YT, YU, YW$ are the tangents to the curves $YY, VV, ZZ$. Moreover, although we may be content with the assignable quantities $(d)v, (d)v, (d)z, (d)x$, etc., since in this way we may perceive the whole fruit of our calculus, namely a construction by means of assignable quantities, yet it is plain from what I have said that, at least in our minds, the unassignables $dx$ and $dy$ may be substituted for them by a method of supposition even in the case when they are evanescent; for the ratio $dy : dx$ can always be reduced to the ratio $(d)y : (d)x$, a ratio between quantities that are assignable or undoubtedly real. Thus we have in the case of tangents $dv : dy = 1 - dz : dx$, or $dv = dy - dz$.

**Multiplication.** Let $ay = xv$, then $a(dy) = x(d)v + v(dx)$.

Proof. $ay + a dy = x + dx, v + dv = xv + x dv + v dx + dx dv$; and, rejecting the equals $ay$ and $xy$ from the two sides,

$$a dy = x dv + v dx + dx dv,$$

or

$$\frac{a dy}{dx} = \frac{x dv}{dx} + v + dv;$$

and transferring the matter, as we may, to straight lines that never become evanescent, we have

$$\frac{a(d)y}{(d)x} + \frac{x(d)y}{(d)x} + v + dv;$$

so that, since it alone can become evanescent, $dv$ is superfluous, and in the case of the vanishing differences, as in the case that $dv = 0$, we have

$$a(d)y + x(d)v + v(d)x, \text{ as was stated},$$

or

$$(d)y : (d)x = x + v : a.$$

Also, since $(d)y : (d)x$ always $= dy : dx$, it will be allowable to suppose this is true in the case when $dy, dx$ become evanescent, and to say that $dy : dx = x + v : a$, or $a dy = x dv + v dx$.

**Division.** Let $z : a = v : x$, then $(d)z : a = v(d)x = x(d)y : xx$.

Proof.

$$z + dz : a = v + dv : x + dx;$$
or clearing of fractions, \(xz + xdz + zdx + dxz = av + adv\); taking away the equals \(xz\) and \(av\) from the two sides, and dividing what is left by \(dx\), we have
\[
a dv = x dz,; \ dx = z + dz,
\]
or
\[
a(d)v - x(d)z,; \ dx = z + dz;
\]
and thus, only \(dz\), which can become evanescent, is superfluous. Also, in the case of vanishing differences, when \(1X\) coincides with \(2X\), since in that case \(dz = 0\), we have
\[
a(d)v - x(d)z,; \ (d)x = z = av : x;
\]
whence, (as was stated) \((d)z + ax(d)v - av(d)x, : xx\), or
\[
(d)z : (d)x = (a : x)(d)v : (d)x - av : xx.
\]
Also, since \((d)z : (d)x\) is always equal to \(dz : dx\), on all other occasions, it is allowable to suppose this to be also when \(dz, dv, dx\) are evanescent, and to put
\[
dz : dx = ax dv - av dx, : xx
\]
For \textit{Powers}, let the equation be \(a^{n-2} x^e = y^n\), then
\[
\frac{(d)y}{(d)x} = \frac{e_x^{e-1}}{n y^{n-1}};
\]
and this I will prove in a manner a little more detailed than those above, thus:
\[
a^{n-e} \cdot \frac{1}{1} x^e + \frac{e}{1} x^{e-2} dx + \frac{e, e - 1}{1, 2} x^{e-2} dx dx + \frac{e, e - 1, e - 2}{1, 2, 3} x^{e-3} dx dx dx
\]
(and so on until the factor \(e - e\) or 0 is reached)
\[
= \frac{1}{1} y^n + \frac{n}{1} y^{n-1} dy + \frac{n, n - 1}{1, 2} y^{n-2} dy dy + \frac{n, n - 1, n - 2}{1, 2, 3} y^{n-3} dy dy dy
\]
(and so on until the factor \(n - n\) or 0 is reached); take away from the one side \(a^{n-2} x^e\) and from the other side \(y^n\), these being equal to one another, and divide what is left by \(dx\), and lastly, instead of the ratio \(dy : dx\), between the two quantities that continually diminish, substitute the ratio that is equal to it, \((d)y : (d)x\), a ratio between two quantities, of which one, \((d)x\), always remains the same during the time that the differences are diminishing, or while \(2X\) is approaching the fixed point \(1X\) and we have
\[
\frac{e}{1} x^{e-1} + \frac{e, e - 1}{1, 2} x^{e-2} dx + \frac{e, e - 1, e - 2}{1, 2, 3} x^{e-3} dx dx + \text{etc.} = \frac{n}{1} y^{n-1} + \frac{n, n - 1}{1, 2} y^{n-2} dy + \frac{n, n - 1, n - 2}{1, 2, 3} y^{n-3} dy dy + \text{etc.}
\]
Now, since by the postulate there is included in this general rule the case also in which the differences become equal to zero, that is when the points 2X, 2Y coincide with the points 1X, 1Y respectively; therefore, in that case, putting $dx$ and $dy$ equal to 0, we have

$$\frac{e}{1^{e-1}}x^{e-1} = n\frac{y^{n-1}(d)y}{(d)x},$$

the remaining terms vanishing, or $(d)y : (d)x = e.x^{e-1} : ny^{n-1}$. Moreover, as we have explained, the ratio $(d)y : (d)x$ is the same as the ratio of $y$, or the ordinate 1X 1Y, to the subtangent 1XT, where it is supposed that $T_1Y$ touches the curve in 1Y.

This proof holds good whether the powers are integral powers or roots of which the exponents are fractions. Though we may also get rid of fractional exponents by raising each side of the equation to some power, so that $e$ and $n$ will then signify nothing else but powers with rational exponents, and there will be no need of a series proceeding to infinity. Moreover, at any rate, it will be permissible, by means of the explanation given above, to return to the unassignable quantities $dy$ and $dx$, by making in the case of evanescent differences, as in all other cases, the supposition that the ratio of the evanescent quantities $dy$ and $dx$ is equal to the ratio of $(d)y$ and $(d)x$, because this supposition can always be reduced to an undoubtable truth.

[[Second Derivatives Follow. Didn’t include these]]
method of fluxions, which I have made use of here in the quadrature of curves, in the years 1665 and 1666.

3. Fluxions are very nearly as the augments of the fluents generated in equal but very small particles of time, and, to speak accurately, they are in the first ratio of the nascent augments; but they may be expounded by any lines which are proportional to them.

4. Thus if the areas \( ABC, ABDG \) be described by the ordinates \( BC, BD \) moving along the base \( AB \) with an uniform motion, the fluxions of these area’s shall be to one another as the describing ordinates \( BC \) and \( BD \), and may be expounded by these ordinates, because that these ordinates are as the nascent augments of the areas.

5. Let the ordinate \( BC \) advance from it’s place into any new place be. Complete the parallelogram \( BCE \), and draw the right line \( VTH \) touching the curve in \( C \), and meeting the two lines \( be \) and \( BA \) produced in \( T \) and \( V \); and \( Be, Ec \) and \( Cc \) will be the augments now generated of the absciss \( AB \), the ordinate \( BC \) and the curve line \( ACc \); and the sides of the triangle \( CET \) are in the first ratio of these augments considered as nascent, therefore the fluxions of \( AB, BC \) and \( AC \) are as the sides \( CE, ET \) and \( CT \) of that triangle \( CET \), and may be expounded by these same sides, or, which is the same thing, by the sides of the triangle \( VBC \), which is similar to the triangle \( CET \).

6. It comes to the same purpose to take the fluxions in the ultimate ratio of the evanescent parts. Draw the right line \( Cc \), and produce it to \( K \). Let the ordinate \( bc \) return into it’s former place \( BC \), and when the points \( C \) and \( c \) coalesce, the right line \( CK \) will coincide with the tangent \( CH \), and the evanescent triangle \( CEc \) in it’s ultimate form will become similar to the triangle \( GET \), and it’s evanescent sides \( CE, Er \) and \( Cc \) will be ultimately among themselves as the sides \( CE, ET \) and \( CT \) of the other triangle \( GET \), are, and therefore the fluxions of the lines \( AB, BC \) and \( AC \) are in the same ratio. If the points \( C \) and \( c \) are distant from one another by any small distance, the right line \( CK \) will likewise be distant from the tangent \( CH \) by a small distance. That the right line \( CK \) may coincide with the tangent \( CH \), and the ultimate ratios of the lines \( CE, Ec \) and \( Cc \) may be found, the points \( C \) and \( c \) ought to coalesce and exactly coincide. The very smallest errors in mathematical matters are not to be neglected.

7. By the like way of reasoning, if a circle described with the center \( B \) and radius \( BC \) be drawn at right angles along the absciss \( AB \), with an uniform motion, the fluxion of the generated solid \( ABC \) will be as that generating circle, and the fluxion of it’s superficies will be as the perimeter of that circle and the fluxion of the curve line \( AC \) jointly. For in whatever time the solid \( ABC \) is generated by drawing that circle along the length of the absciss, in the same time it’s superficies is generated by drawing the perimeter of that circle along the length of the curve \( AC \). ...

11. Let the quantity \( x \) flow uniformly, and let it be proposed to find the fluxion of \( x^n \).
In the same time that the quantity $x$ by flowing, becomes $x - o$, the quantity $x^n$ will become $(x + o)^n$, that is, by the method of infinite series, $x^n + nox^{n-1} + \frac{n^2-n}{2}oxx^{n-1} + \&c$. And the augments $o$ and $nox^{n-1} + \frac{n^2-n}{2}oxx^{n-1} + \&c$ are to one another as $1$ and $nx^{n-1} + \frac{n^2-n}{2}oxx^{n-1} + \&c$

Now let these augments vanish, and their ultimate ratio will be $1$ to $nx^{n-1}$.

12. By like ways of reasoning, the fluxions of lines, whether right or curve in all cases, as likewise the fluxions of superficies angles and other quantities, may be collected by the method of prime and ultimate ratios. Now to institute an analysis after this manner in finite quantities and investigate the prime or ultimate ratios of these finite quantities when in their nascent or evanescent state, is consonant to the geometry of the ancients: and I was willing to show that, in the method of fluxions, there is no necessity of introducing figures infinitely small into geometry. Yet the analysis may be performed in any kind of figures, whether finite or infinitely small, which are imagined similar to the evanescent figures; as likewise in these figures, which, by the method of indivisibles, used to be reckoned as infinitely small, provided you proceed with due caution.

...  

PROPOSITION I. PROBLEM I

15. An equation being given involving any number of flowing quantities, to find the fluxions.

Solution. Let every term of the equation be multiplied by the index of the power of every flowing quantity that it involves, and in every multiplication change the side or root of the power into its fluxion, and the aggregate of all the products with their proper signs, will be the new equation.

16. Explication. Let $a$, $b$, $c$, $d$, &c. be determinate and invariable quantities, and let any equation be proposed involving the flowing quantities $x$, $y$, &c. as $x^3 - xy^2 + a^2z - b^3 = 0$. Let the terms be first multiplied by the indexes of the power of $x$, and in every multiplication for the root, or $x$ of one dimension write $\dot{x}$, and the sum of the factors will be $3\dot{x}x^2 - xy^2$. Do the same in $y$ and there arises $-2\dot{xy}\dot{y}$. Do the same in $z$, and there arises $aa\dot{z}$. Let the sum of these products be put equal to nothing, and you’ll have the equation $3\dot{x}x^2 - \dot{y}y^2 - 2\dot{xy}y + aa\dot{z} = 0$. I say the relation of the fluxions is defined by this equation.

17. Demonstration. For let $o$ be a very small quantity, and let $0\dot{z}$, $o\dot{y}$, $o\dot{x}$ be the moments, that is the momentaneous synchronal increments of the quantities $z, y, x$. And if the flowing quantities are just now $z, y, x$, then after a moment of time, being increased by their increments $0\dot{z}$, $o\dot{y}$, $o\dot{x}$, these quantities shall become $z + 0\dot{z}$, $y + o\dot{y}$, $x + o\dot{x}$: which being wrote in the first equation for $z, y$ and $x$, give this equation $x^3 + 3x^2o\dot{x} + 3xoo\dot{x}\dot{x} + o^3\dot{x}^3 - xy^2 - o\dot{xy}^2 - 2xo\dot{y}y - 2\dot{x}x^2\dot{y}y - xo^2\dot{y}y - xo\dot{y}^2\dot{y} + a^2z + a^2o\dot{z} - b^3 = 0$.

Subtract the former equation from the latter, divide the remaining equation by $o$, and it will be $e\dot{x}x^2 + 3\dot{x}xox + x^3o^2 - \dot{x}y^2 - 2\dot{xy}y - 2xo\dot{y}y - xo\dot{y}^2 - xo\dot{y}^2 + a^2\dot{x} = 0$. Let the quantity $o$ be diminished infinitely, and neglecting the terms
which vanish, there will remain \(3\dot{x}x - \dot{y}y^2 - 2x\dot{y}y + a^2\dot{z} = 0\). Q.E.D.

18. A fuller explication. After the same manner if the equation were \(x^3 - xy^2 + ax\sqrt{ax - y^2} - b^3 = 0\), thence would be produced \(3x^2\dot{x} - \dot{y}y^2 - 2x\dot{y}y + ax\sqrt{ax - y^2} = 0\). Where if you would take away the fluxion \(\sqrt{ax - y^2}\), put \(\sqrt{ax - y^2} = z\), and it will be \(ax - y^2 = z^2\), and by this proposition \(a\ddot{x} - 2\dot{y}y = 2\dot{z}z\), or \(\frac{a\ddot{x} - 2\dot{y}y}{2\sqrt{ax - y^2}} = \dot{z}\), that is \(\frac{a\ddot{x} - 2\dot{y}y}{2\sqrt{ax - y^2}} = \sqrt{ax - y^2}\). And thence \(3x^2\dot{x} - \dot{y}y^2 = 2x\dot{y}y + \frac{a^2\ddot{x} - 2\dot{y}y}{2\sqrt{ax - y^2}} = 0\).

19. And by repeating the operation, you proceed to second, third, ad subseuent fluxions. Let \(zy^3 + z^4 + a^4 = 0\) be an equation proposed, and by the first operation it becomes \(\dot{z}y^3 + 3\dot{y}yy^2 - 4\dot{z}z^3 = 0\); by the second \(\dot{z}y^3 + 6\dot{y}yy^2 + 3\dot{z}yy^2 + 6\dot{y}yy^2 - 4\dot{z}z^3 = 0\), by the third, \(\dot{z}y^3 + 9\dot{z}yy^2 + 18\dot{y} yy^2 + 3\dot{z}yy^2 + 18\ddot{y}yy + 3\ddot{y}yy + 6\dot{z}y^3 - 4\dot{z}z^3 + 36\dot{z}z^2 = 0\).

20. But when one proceeds thus to second, third, and following fluxions, it is proper to consider some quantity as flowing uniformly, and for its first fluxion to write unity, for the second and subsequent ones, nothing. Let there be given the equation \(zy^3 - z^4 + a^4 = 0\), as above; and let \(z\) flow uniformly, and let its fluxion be unity: then by the first operation it shall be \(y^3 + 3\dot{z}yy^2 - 4\dot{z}z^3 = 0\); by the second \(6\dot{y}yy^2 + 3\dot{z}yy^2 + 6\dot{y}yy^2 - 12\dot{z}y^2 = 0\); by the third \(9\ddot{y}yy + 18\dot{y} yy^2 + 3\dot{z}yy^2 + 18\ddot{y}yy + 6\dot{z}y^3 - 24\dot{z}z = 0\).

But in equations of this kind it must be conceived that the fluxions in all the terms are of the same order, i.e., either all of the first order \(\ddot{y}, \dot{z}\); or all of the second \(\dot{y}, \dot{y}^2, \dot{y}z, \dot{z}^2\); or all of the third \(\ddot{y}, \ddot{y}y, \ddot{y}z, \dot{y}^2z, \dot{z}^3\), &c. And where the case is otherwise the order is to be completed by means of the fluxions of a quantity that flows uniformly, which fluxions are understood. Thus the last equation, by completing the third order, becomes \(9\ddot{y}yy + 18\dot{y} yy^2 + 3\dot{z}yy^2 + 18\ddot{y}yy + 6\dot{z}y^3 - 24\dot{z}z = 0\).

... PROPOSITION I. PROBLEM I 15. An equation being given involving any number of flowing quantities, o find the fluxions. Solution. Let every term of the equation be multiplied by the index of the power of every flowing quantity that it involves, and in every multiplication change the side or root of the power into its fluxion, and the aggregate of all the products with their proper signs, will be the new equation.

16. Explication. Let \(a, b, c, d, &c.\) be determinate and invariable quantities, and let any equation be proposed involving the flowing quantities \(x, y, &c.\) as \(x^3 - xy^2 + a^2z - b^3 = 0\). Let the terms be first multiplied by the indexes of the power of \(x\), and in every multiplication for the root, or \(x\) of one dimension write \(\dot{x}\), and the sum of the factors will be \(3\dot{x}x^2 - \dot{y}y^2\). Do the same in \(y\) and there arises \(-2xy\ddot{y}\). Do the same in \(z\), and there arises \(a\ddot{z}\). Let the sum of these products be put equal to nothing, and you’ll have the equation \(3\dot{x}x^2 - \dot{y}y^2 - 2xy\ddot{y} + a\ddot{z} = 0\). I say the relation of the fluxions is defined by this equation.

17. Demonstration. For let \(o\) be a very small quantity, and let \(0\dot{z}, o\dot{y}, o\dot{x}\) be
the moments, that is the momentaneous synchronal increments of the quantities $z, y, x$. And if the flowing quantities are just now $z, y, x$, then after a moment of time, being increased by their increments $O\dot{z}, o\dot{y}, o\dot{x}$, these quantities shall become $z + O\dot{z}, y + o\dot{y}, x + o\dot{x}$: which being wrote in the first equation for $z, y$ and $x$, give this equation $x^3 + 3x^2O\dot{x} + 3xo^2O\dot{x} = a^3\dot{z}^3 - xy^2 - o\dot{x}y^2 - 2xO\dot{y}y - 2xo^2\dot{y}y - xo^2\dot{y}y - xo^2\dot{y}y + a^2z + a^2o\dot{z} - b^3 = 0$.

Subtract the former equation from the latter, divide the remaining equation by $o$, and it will be $e\ddot{x}x^2 + 3x\dot{x}o + x^3a^2 - \dot{x}y^2 - 2\dot{x}O\dot{y}y - 2xO\dot{y}y - xo\dot{y}y - xo\dot{y}y + a^2\dot{x} = 0$. Let the quantity $o$ be diminished infinitely, and neglecting the terms which vanish, there will remain $3\dot{x}x - \dot{y}y^2 - 2xy + a^2\dot{z} = 0$. Q.E.D.

18. A fuller explication. After the same manner if the equation were $x^3 - x\dot{y}y^2 + 3a\sqrt{ax - y^2} - b^3 = 0$, thence would be produced $3x\dot{x}x - \dot{y}y^2 - 2xy + 3a\sqrt{ax - y^2} = 0$. Where if you would take away the fluxion $\dot{y}$, put $\sqrt{ax - y^2} = z$, and it will be $ax - y^2 = z^2$, and by this proposition $a\dot{x} - 2\dot{y}y = 2\dot{z}z$, or $\frac{a\dot{x} - 2\dot{y}y}{2\dot{z}z} = \dot{z}$, that is $\frac{a\dot{x} - 2\dot{y}y}{2\sqrt{ax - y^2}} = \sqrt{ax - y^2}$. And thence $3x\dot{x}x - \dot{y}y^2 = 2xy + \frac{a^3 - 2\dot{y}y}{2\sqrt{ax - y^2}} = 0$.

19. And by repeating the operation, you proceed to second, third, ad subsequent fluxions. Let $zy^3 + z^4 + a^4 = 0$ be an equation proposed, and by the first operation it becomes $\ddot{z}y^3 + 3z\dot{y}y^2 = 4z\dot{z}z^3 = 0$; by the second

$\ddot{z}y^3 + 6\dot{z}\dot{y}y^2 + 3z\dot{y}y^2 + 6z\dot{y}^2y - 4\ddot{z}z^3 - 12\dot{z}\dot{z}z^2 = 0$, by the third, $\dddot{z}y^3 + 9\ddot{z}\dot{y}y^2 + 18\dot{z}\dot{y}y^2 + 18z\ddot{y}y + 6z\dot{y}^2y = 4\dddot{z}z^3 - 36\ddot{z}\dot{z}z^2 - 24\dot{z}\dot{z}z = 0$.

20. But when one proceeds thus to second, third, and following fluxions, it is proper to consider some quantity as flowing uniformly, and for its first fluxion to write unity, for the second and subsequent ones, nothing. Let there be given the equation $zy^3 - z^4 + a^4 = 0$, as above; and let $z$ flow uniformly, and let its fluxion be unity: then by the first operation it shall be $y^3 + 3z\dot{y}y^2 - 4z\dot{z}z^3 = 0$; by the second $6\ddot{y}y + 3z\dot{y}y^2 + 6z\dot{y}^2y - 24\dot{z}z^2 = 0$; by the third $9\ddot{y}y + 18\dot{y}y^2 + 3z\dot{y}y^2 + 18z\ddot{y}y + 6z\dot{y}^2y = 18z\ddot{y}y + 6z\dot{y}^3 - 24\dot{z}z^3 = 0$.

But in equations of this kind it must be conceived that the fluxions in all the terms are of the same order, i.e., either all of the first order $\dot{y}, \dot{z}$; or all of the second $\ddot{y}, \ddot{z}$; or all of the third $\dddot{y}, \dddot{z}$, &c. And where the case is otherwise the order is to be completed by means of the fluxions of a quantity that flows uniformly, which fluxions are understood. Thus the last equation, by completing the third order, becomes $9z\dddot{y}y + 18z\ddot{y}^2y + 3z\dddot{y}y^2 + 18z\dddot{y}y + 6z\dot{y}^3 - 24z\dot{z}z^3 = 0$.

7.6 Leibniz. FTC. 1693.

I shall now show that the general problem of quadratures can be reduced to the nding of a line that has a given law of tangency (declivitas), that is, for which the sides of the characteristic triangle have a given mutual relation. Then I shall show how this line can be described by a motion that I have invented. For
this purpose [Figure 3.11] I assume for every curve \( C(C') \) a double characteristic triangle, one, \( TBC \), that is assignable, and one, \( GLC \), that is inassignable, and these two are similar. The inassignable triangle consists of the parts \( GL, LC \), with the elements of the coordinates \( CF, CB \) as sides, and \( GC \), the element of arc, as the base or hypotenuse. But the assignable triangle \( TBC \) consists of the axis, the ordinate, and the tangent, and therefore contains the angle between the direction of the curve (or its tangent) and the axis or base, that is, the inclination of the curve at the given point \( C \). Now let \( F(H) \), the region of which the area has to be squared, be enclosed between the curve \( H(H) \), the parallel lines \( FH \) and \( (F)(H) \), and the axis \( F(F) \); on that axis let \( A \) be a fixed point, and let a line \( AB \), the conjugate axis, be drawn through \( A \) perpendicular to \( AF \). We assume that point \( C \) lies on \( HF \) (continued if necessary); this gives a new curve \( C(C') \) with the property that, if from point \( C \) to the conjugate axis \( AB \) (continued if necessary) both its ordinate \( CB \) (equal to \( AF \) ) and tangent \( CT \) are drawn, the part \( TB \) of the axis between them is to \( BC \) as \( HF \) to a constant [segment] \( a \), or \( a \) times \( BT \) is equal to the rectangle \( AFH \) (circumscribed about the trilinear figure \( AFHA \)). This being established, I claim that the rectangle on \( a \) and \( E(C) \) (we must discriminate between the ordinates \( FC \) and \( (F)(C) \) of the curve) is equal to the region \( F(H) \). When therefore I continue line \( H(H) \) to \( A \), the trilinear figure \( AFHA \) of the figure to be squared is equal to the rectangle with the constant \( a \) and the ordinate \( FC \) of the squaring curve as sides. This follows immediately from our calculus. Let \( AF = y, FH = z, BT = t \), and \( FC = x \); then \( t = zy : a \), according to our assumption; on the other hand, \( t = ydx : dy \) because of the property of the tangents expressed in our calculus. Hence \( adx = zdy \) and therefore \( ax = \int zdy = AFHA \). Hence the curve \( C(C') \) is the quadratrix with respect to the curve \( H(H) \), while the ordinate \( FC \) of \( C(C') \), multiplied by the constant \( a \), makes the rectangle equal to the area, or the sum of the ordinates \( H(H) \) corresponding to the corresponding abscissas \( AF \). Therefore, since \( BT : AF = FH : a \) (by assumption), and the relation of this \( FH \) to \( AF \) (which expresses the nature of the figure to be squared) is given, the relation of \( BT \) to \( FH \) or to \( BC \), as well as that of \( BT \) to \( TC \), will be given, that is, the relation between the sides of triangle \( TBC \). Hence, all that is needed to be able to perform the quadratures and measurements is to be able to describe the curve \( C(C') \) (which, as we have shown, is the quadratrix), when the relation between the sides of the assignable characteristic triangle \( TBC \) (that is, the law of inclination of the curve) is given.

\[ \text{[ref 41]} \]
7.7 Leibniz, on infinitesimals, 1701?

[ref 42, 42b]

7.8 Newton, comments on infinity. 1690?

"There are different degrees of infinity or of infinitely small, just as the globe of the Earth is estimated as a point in proportion to the distance of the fixed stars, and a play ball is still a point as compared to the radius of the terrestrial sphere, so that the distance of the fixed starts is an infinitely infinite or infinite of the infinite with respect to the diameter of the ball"

Memoir of Mr. G. G. Leibniz concerning his feelings about differential calculus.

Item 42B.

[ref 43]
Chapter 8

Foundations and the ’Modern’ calculus

8.1 Berkeley’s objections. 1734

[ref 44] http://ia361301.us.archive.org/7/items/theanalystoradis00berkuoft/theanalystoradis00berkuoft.pdf

THE ANALYST;
OR, A
DISCOURSE
Addressed to an
Infidel MATHEMATICIAN.
WHEREIN
It is examined whether the Object, Principles, and Inferences of the modern Analysis are more distinctly conceived, or more evidently deduced, than Religious Mysteries and Points of Faith.

by the author of
THE MINUTE PHILOSOPHER

‘First cast out the beam out of thine own eye; and then shalt thou see clearly to cast out the mote out of thy brother’s eye.’
MATT. c. vii. v 5

First published in 1734

I. Though I am a Stranger to your Person, yet I am not, Sir, a Stranger to the Reputation you have acquired, in that branch of Learning which hath
been your peculiar Study; nor to the Authority that you therefore assume in things foreign to your Profession, nor to the Abuse that you, and too many more of the like Character, are known to make of such undue Authority, to the misleading of unwary Persons in matters of the highest Concernment, and whereof your mathematical Knowledge can by no means qualify you to be a competent Judge. Equity indeed and good Sense would incline one to disregard the Judgment of Men, in Points which they have not considered or examined. But several who make the loudest Claim to those Qualities, do, nevertheless, the very thing they would seem to despise, clothing themselves in the Livery of other Mens Opinions, and putting on a general deference for the Judgment of you, Gentlemen, who are presumed to be of all Men the greatest Masters of Reason, to be most conversant about distinct Ideas, and never to take things on trust, but always clearly to see your way, as Men whose constant Employment is the deducing Truth by the justest inference from the most evident Principles. With this bias on their Minds, they submit to your Decisions where you have no right to decide. And that this is one short way of making Infidels I am credibly informed.

II. Whereas then it is supposed, that you apprehend more distinctly, consider more closely, infer more justly, conclude more accurately than other Men, and that you are therefore less religious because more judicious, I shall claim the privilege of a Free-Thinker; and take the Liberty to inquire into the Object, Principles, and Method of Demonstration admitted by the Mathematicians of the present Age, with the same freedom that you presume to treat the Principles and Mysteries of Religion; to the end, that all Men may see what right you have to lead, or what Encouragement others have to follow you. It hath been an old remark that Geometry is an excellent Logic. And it must be owned, that when the Definitions are clear; when the Postulata cannot be refused, nor the Axioms denied; when from the distinct Contemplation and Comparison of Figures, their Properties are derived, by a perpetual well-connected chain of Consequences, the Objects being still kept in view, and the attention ever fixed upon them; there is acquired a habit of reasoning, close and exact and methodical: which habit strengthens and sharpens the Mind, and being transferred to other Subjects, is of general use in the inquiry after Truth. But how far this is the case of our Geometrical Analysts, it may be worth while to consider.

III. The Method of Fluxions is the general Key, by help whereof the modern Mathematicians unlock the secrets of Geometry, and consequently of Nature. And as it is that which hath enabled them so remarkably to outgo the Ancients in discovering Theorems and solving Problems, the exercise and application thereof is become the main, if not sole, employment of all those who in this Age pass for profound Geometers. But whether this Method be clear or obscure, consistent or repugnant, demonstrative or precarious, as I shall inquire with the utmost impartiality, so I submit my inquiry to your own Judgment, and that of every candid Reader. Lines are supposed to be generated [NOTE: Introd. ad Quadraturam Curvarum.] by the motion of Points, Planes by the motion of Lines, and Solids by the motion of Planes. And whereas Quantities generated
in equal times are greater or lesser, according to the greater or lesser Velocity, wherewith they increase and are generated, a Method hath been found to determine Quantities from the Velocities of their generating Motions. And such Velocities are called Fluxions: and the Quantities generated are called flowing Quantities. These Fluxions are said to be nearly as the Increments of the flowing Quantities, generated in the least equal Particles of time; and to be accurately in the first Proportion of the nascent, or in the last of the evanescent, Increments. Sometimes, instead of Velocities, the momentaneous Increments or Decrements of undetermined flowing Quantities are considered, under the Appellation of Moments.

IV. By Moments we are not to understand finite Particles. These are said not to be Moments, but Quantities generated from Moments, which last are only the nascent Principles of finite Quantities. It is said, that the minutest Errors are not to be neglected in Mathematics: that the Fluxions are Celerities, not proportional to the finite Increments though ever so small; but only to the Moments or nascent Increments, whereof the Proportion alone, and not the Magnitude, is considered. And of the aforesaid Fluxions there be other Fluxions, which Fluxions of Fluxions are called second Fluxions. And the Fluxions of these second Fluxions are called third Fluxions: and so on, fourth, fifth, sixth, &c. ad infinitum. Now as our Sense is strained and puzzled with the perception of Objects extremely minute, even so the Imagination, which Faculty derives from Sense, is very much strained and puzzled to frame clear Ideas of the least Particles of time, or the least Increments generated therein: and much more so to comprehend the Moments, or those Increments of the flowing Quantities in statu nascenti, in their very first origin or beginning to exist, before they become finite Particles. And it seems still more difficult, to conceive the abstracted Velocities of such nascent imperfect Entities. But the Velocities of the Velocities, the second, third, fourth, and fifth Velocities, &c. exceed, if I mistake not, all Humane Understanding. The further the Mind analyseth and pursueth these fugitive Ideas, the more it is lost and bewildered; the Objects, at first fleeting and minute, soon vanishing out of sight. Certainly in any Sense a second or third Fluxion seems an obscure Mystery. The incipient Celerity of an incipient Celerity, the nascent Augment of a nascent Augment, i. e. of a thing which hath no Magnitude: Take it in which light you please, the clear Conception of it will, if I mistake not, be found impossible, whether it be so or no I appeal to the trial of every thinking Reader. And if a second Fluxion be inconceivable, what are we to think of third, fourth, fifth Fluxions, and so onward without end?

V. The foreign Mathematicians are supposed by some, even of our own, to proceed in a manner, less accurate perhaps and geometrical, yet more intelligible. Instead of flowing Quantities and their Fluxions, they consider the variable finite Quantities, as increasing or diminishing by the continual Addition or Subduction of infinitely small Quantities. Instead of the Velocities whereby Increments are generated, they consider the Increments or Decrements themselves, which they call Differences, and which are supposed to be infinitely small. The Difference of a Line is an infinitely little Line; of a Plane an infinitely
little Plane. They suppose finite Quantities to consist of Parts infinitely little, and Curves to be Polygons, whereof the Sides are infinitely little, which by the Angles they make one with another determine the Curvity of the Line. Now to conceive a Quantity infinitely small, that is, infinitely less than any sensible or imaginable Quantity, or any the least finite Magnitude, is, I confess, above my Capacity. But to conceive a Part of such infinitely small Quantity, that shall be still infinitely less than it, and consequently though multiply’d infinitely shall never equal the minutest finite Quantity, is, I suspect, an infinite Difficulty to any Man whatsoever; and will be allowed such by those who candidly say what they think; provided they really think and reflect, and do not take things upon trust.

VI. And yet in the calculus differentialis, which Method serves to all the same Intents and Ends with that of Fluxions, our modern Analysts are not content to consider only the Differences of finite Quantities: they also consider the Differences of those Differences, and the Differences of the Differences of the first Differences. And so on ad infinitum. That is, they consider Quantities infinitely less than the least discernible Quantity; and others infinitely less than those infinitely small ones; and still others infinitely less than the preceding Infinitesimals, and so on without end or limit. Insomuch that we are to admit an infinite succession of Infinitesimals, each infinitely less than the foregoing, and infinitely greater than the following. As there are first, second, third, fourth, fifth &c. Fluxions, so there are Differences, first, second, third fourth, &c. in an infinite Progression towards nothing, which you still approach and never arrive at. And (which is most strange) although you should take a Million of Millions of these Infinitesimals, each whereof is supposed infinitely greater than some other real Magnitude, and add them to the least given Quantity, it shall be never the bigger. For this is one of the modest postulata of our modern Mathematicians, and is a Corner-stone or Ground-work of their Speculations.

VII. All these Points, I say, are supposed and believed by certain rigorous Ex-actors of Evidence in Religion, Men who pretend to believe no further than they can see. That Men, who have been conversant only about clear Points, should with difficulty admit obscure ones might not seem altogether unaccountable. But he who can digest a second or third Fluxion, a second or third Difference, need not, methinks, be squeamish about any Point in Divinity. There is a nat-ural Presumption that Mens Faculties are made alike. It is on this Supposition that they attempt to argue and convince one another. What, therefore, shall appear evidently impossible and repugnant to one, may be presumed the same to another. But with what appearance of Reason shall any Man presume to say, that Mysteries may not be Objects of Faith, at the same time that he himself admits such obscure Mysteries to be the Object of Science?

VIII. It must indeed be acknowledged, the modern Mathematicians do not consider these Points as Mysteries, but as clearly conceived and mastered by their comprehensive Minds. They scruple not to say, that by the help of these new Analytics they can penetrate into Infinity it self: That they can even ex- tend their Views beyond Infinity: that their Art comprehends not only Infinite,
but Infinite of Infinite (as they express it) or an Infinity of Infinites. But, notwithstanding all these Assertions and Pretensions, it may be justly questioned whether, as other Men in other Inquiries are often deceived by Words or Terms, so they likewise are not wonderfully deceived and deluded by their own peculiar Signs, Symbols, or Species. Nothing is easier than to devise Expressions or Notations for Fluxions and Infinitesimals of the first, second, third, fourth, and subsequent Orders, proceeding in the same regular form without end or limit . . . . &c. or \( dx \), \( ddx \), \( dddx \), \( dddddx \). &c. These Expressions indeed are clear and distinct, and the Mind finds no difficulty in conceiving them to be continued beyond any assignable Bounds. But if we remove the Veil and look underneath, if laying aside the Expressions we set ourselves attentively to consider the things themselves, which are supposed to be expressed or marked thereby, we shall discover much Emptiness, Darkness, and Confusion; nay, if I mistake not, direct Impossibilities and Contradictions. Whether this be the case or no, every thinking Reader is intreated to examine and judge for himself.

IX. Having considered the Object, I proceed to consider the Principles of this new Analysis by Momentums, Fluxions, or Infinitesimals; wherein if it shall appear that your capital Points, upon which the rest are supposed to depend, include Error and false Reasoning; it will then follow that you, who are at a loss to conduct your selves, cannot with any decency set up for guides to other Men. The main Point in the method of Fluxions is to obtain the Fluxion or Momentum of the Rectangle or Product of two indeterminate Quantities. Inasmuch as from thence are derived Rules for obtaining the Fluxions of all other Products and Powers; be the Coefficients or the Indexes what they will, integers or fractions, rational or surd. Now this fundamental Point one would think should be very clearly made out, considering how much is built upon it, and that its Influence extends throughout the whole Analysis. But let the Reader judge. This is given for Demonstration. [NOTE: Naturalis Philosophi principia mathematica, l. 2. lem. 2.] Suppose the Product or Rectangle \( AB \) increased by continual Motion: and that the momentaneous Increments of the Sides \( A \) and \( B \) are \( a \) and \( b \). When the Sides \( A \) and \( B \) were deficient, or lesser by one half of their Moments, the Rectangle was

\[ AB, \quad \text{i. e.,} \quad . \]

And as soon as the Sides \( A \) and \( B \) are increased by the other two halves of their Moments, the Rectangle becomes

\[ AB + aB + bA + ab \]

or . From the latter Rectangle subduct the former, and the remaining Diff-

\[ aB + bA + ab \]

Therefore the Increment of the Rectangle generated by the intire Increments \( a \) and \( b \) is \( aB + bA \). Q.E.D. But it is plain that the direct and true Method to obtain the Moment or Increment of the Rectangle \( AB \), is to take the Sides as increased by their whole Increments, and so multiply them together, \( A + a \) by \( B + b \), the Product whereof \( AB + aB + bA + ab \) is the augmented Rectangle; whence if we subduct \( AB \), the Remainder \( aB + bA + ab \) will be the true Increment of the Rectangle, exceeding that which was obtained by the former illegitimate and indirect Method by the Quantity \( ab \). And this holds universally be the Quantities \( a \) and \( b \) what they will, big or little, Finite or Infinitesimal, Increments, Moments, or Velocities. Nor will it avail to say that
ab is a Quantity exceeding small: Since we are told that in rebus mathematicis
errores qum minimi non sunt contemnendi. [NOTE: Introd. ad Quadraturam
Curvarum.]

X. Such reasoning as this for Demonstration, nothing but the obscurity of the
Subject could have encouraged or induced the great Author of the Fluxionary
Method to put upon his Followers, and nothing but an implicit deference to
Authority could move them to admit. The Case indeed is difficult. There can
be nothing done till you have got rid of the Quantity \(ab\). In order to this the
Notion of Fluxions is shifted: it is placed in various Lights: Points which should
be as clear as first Principles are puzzled; and Terms which should be steadily
used are ambiguous. But notwithstanding all this address and skill the point
of getting rid of \(ab\) cannot be obtained by legitimate reasoning. If a Man by
Methods, not geometrical or demonstrative, shall have satisfied himself of the
usefulness of certain Rules; which he afterwards shall propose to his Disciples
for undoubted Truths; which he undertakes to demonstrate in a subtile manner,
and by the help of nice and intricate Notions; it is not hard to conceive that
such his Disciples may, to save themselves the trouble of thinking, be inclined
to confound the usefulness of a Rule with the certainty of a Truth, and accept
the one for the other; especially if they are Men accustomed rather to compute
than to think; earnest rather to go on fast and far, than solicitous to set out
warily and see their way distinctly.

XI. The Points or meer Limits of nascent Lines are undoubtedly equal, as
having no more magnitude one than another, a Limit as such being no Quantity.
If by a Momentum you mean more than the very initial Limit, it must be either
a finite Quantity or an Infinitesimal. But all finite Quantities are expressly
excluded from the Notion of a Momentum. Therefore the Momentum must be an
Infinitesimal. And indeed, though much Artifice hath been employ’d to escape
or avoid the admission of Quantities infinitely small, yet it seems ineffectual. For
ought I see, you can admit no Quantity as a Medium between a finite Quantity
and nothing, without admitting Infinitesimals. An Increment generated in a
finite Particle of Time, is it self a finite Particle; and cannot therefore be a
Momentum. You must therefore take an Infinitesimal Part of Time wherein
to generate your Momentum. It is said, the Magnitude of Moments is not
considered: And yet these same Moments are supposed to be divided into Parts.
This is not easy to conceive, no more than it is why we should take Quantities
less than \(A\) and \(B\) in order to obtain the Increment of \(AB\), of which proceeding
it must be owned the final Cause or Motive is very obvious; but it is not so
obvious or easy to explain a just and legitimate Reason for it, or shew it to be
Geometrical.

XII. From the foregoing Principle so demonstrated, the general Rule for
finding the Fluxion of any Power of a flowing Quantity is derived. [NOTE:
Philosophi naturalis principia Mathematica, lib. 2. lem. 2.] But, as there seems
to have been some inward Scruple or Consciousness of defect in the foregoing
Demonstration, and as this finding the Fluxion of a given Power is a Point of pri-
mary Importance, it hath therefore been judged proper to demonstrate the same
8.1. BERKELEY’S OBJECTIONS. 1734

in a different manner independent of the foregoing Demonstration. But whether this other Method be more legitimate and conclusive than the former, I proceed now to examine; and in order thereto shall premise the following Lemma. “If with a View to demonstrate any Proposition, a certain Point is supposed, by virtue of which certain other Points are attained; and such supposed Point be it self afterwards destroyed or rejected by a contrary Supposition; in that case, all the other Points, attained thereby and consequent thereupon, must also be destroyed and rejected, so as from thence forward to be no more supposed or applied in the Demonstration.” This is so plain as to need no Proof.

XIII. Now the other Method of obtaining a Rule to find the Fluxion of any Power is as follows. Let the Quantity \(x\) flow uniformly, and be it proposed to find the Fluxion of \(x^n\). In the same time that \(x\) by flowing becomes \(x + o\), the Power \(x^n\) becomes, i.e. by the Method of infinite Series

\[
\text{and the Increments}
\]

are to one another as

Let now the Increments vanish, and their last Proportion will be \(1 \text{ to } nx^{n-1}\). But it should seem that this reasoning is not fair or conclusive. For when it is said, let the Increments vanish, i.e. let the Increments be nothing, or let there be no Increments, the former Supposition that the Increments were something, or that there were Increments, is destroyed, and yet a Consequence of that Supposition, i.e. an Expression got by virtue thereof, is retained. Which, by the foregoing Lemma, is a false way of reasoning. Certainly when we suppose the Increments to vanish, we must suppose their Proportions, their Expressions, and every thing else derived from the Supposition of their Existence to vanish with them.

XIV. To make this Point plainer, I shall unfold the reasoning, and propose it in a fuller light to your View. It amounts therefore to this, or may in other Words be thus expressed. I suppose that the Quantity \(x\) flows, and by flowing is increased, and its Increment I call \(o\), so that by flowing it becomes \(x + o\). And as \(x\) increaseth, it follows that every Power of \(x\) is likewise increased in a due Proportion. Therefore as \(x\) becomes \(x + o\), \(x^n\) will become that is, according to the Method of infinite Series,

And if from the two augmented Quantities we subduct the Root and the Power respectively, we shall have remaining the two Increments, to wit,

which are therefore Exponents of the Ratio of the Increments. Hitherto I have supposed that \(x\) flows, that \(x\) hath a real Increment, that \(o\) is something. And I have proceeded all along on that Supposition, without which I should not have been able to have made so much as one single Step. From that Supposition it is that I get at the Increment of \(x^n\), that I am able to compare it with the Increment of \(x\), and that I find the Proportion between the two Increments. I now beg leave to make a new Supposition contrary to the first, i.e. I will suppose that there is no Increment of \(x\), or that \(o\) is nothing; which second
Supposition destroys my first, and is inconsistent with it, and therefore with every thing that supposeth it. I do nevertheless beg leave to retain $n x^{n-1}$, which is an Expression obtained in virtue of my first Supposition, which necessarily presupposeth such Supposition, and which could not be obtained without it: All which seems a most inconsistent way of arguing, and such as would not be allowed of in Divinity.

XV. Nothing is plainer than that no just Conclusion can be directly drawn from two inconsistent Suppositions. You may indeed suppose any thing possible: But afterwards you may not suppose any thing that destroys what you first supposed. Or if you do, you must begin de novo. If therefore you suppose that the Augments vanish, i. e. that there are no Augments, you are to begin again, and see what follows from such Supposition. But nothing will follow to your purpose. You cannot by that means ever arrive at your Conclusion, or succeed in, what is called by the celebrated Author, the Investigation of the first or last Proportions of nascent and evanescent Quantities, by instituting the Analysis in finite ones. I repeat it again: You are at liberty to make any possible Supposition: And you may destroy one Supposition by another: But then you may not retain the Consequences, or any part of the Consequences of your first Supposition so destroyed. I admit that Signs may be made to denote either any thing or nothing: And consequently that in the original Notation $x + o$, $o$ might have signified either an Increment or nothing. But then which of these soever you make it signify, you must argue consistently with such its Signification, and not proceed upon a double Meaning: which to do were a manifest Sophism. Whether you argue in Symbols or in Words, the Rules of right Reason are still the same. Nor can it be supposed, you will plead a Privilege in Mathematics to be exempt from them.

XVI. If you assume at first a Quantity increased by nothing, and in the Expression $x + o$, $o$ stands for nothing, upon this Supposition as there is no Increment of the Root, so there will be no Increment of the Power; and consequently there will be none except the first, of all those Members of the Series constituting the Power of the Binomial: you will therefore never come at your Expression of a Fluxion legitimately by such Method. Hence you are driven into the fallacious way of proceeding to a certain Point on the Supposition of an Increment, and then at once shifting your Supposition to that of no Increment. There may seem great Skill in doing this at a certain Point or Period. Since if this second Supposition had been made before the common Division by $o$, all had vanished at once, and you must have got nothing by your Supposition. Whereas by this Artifice of first dividing, and then changing your Supposition, you retain 1 and $n x^{n-1}$. But, notwithstanding all this address to cover it, the fallacy is still the same. For whether it be done sooner or later, when once the second Supposition or Assumption is made, in the same instant the former Assumption and all that you got by it is destroyed, and goes out together. And this is universally true, be the Subject what it will, throughout all the Branches of humane Knowledge; in any other of which, I believe, Men would hardly admit such a reasoning as this, which in Mathematics is accepted for Demonstration.
XVII. It may not be amiss to observe, that the Method for finding the Fluxion of a Rectangle of two flowing Quantities, as it is set forth in the Treatise of Quadratures, differs from the abovementioned taken from the second Book of the Principles, and is in effect the same with that used in the calculus differentialis. [NOTE: Analyse des Infiniment Petits, part 1. prop. 2.] For the supposing a Quantity infinitely diminished and therefore rejecting it, is in effect the rejecting an Infinitesimal; and indeed it requires a marvellous sharpness of Discernment, to be able to distinguish between evanescent Increments and infinitesimal Differences. It may perhaps be said that the Quantity being infinitely diminished becomes nothing, and so nothing is rejected. But according to the received Principles it is evident, that no Geometrical Quantity, can by any division or subdivision whatsoever be exhausted, or reduced to nothing. Considering the various Arts and Devices used by the great author of the Fluxionary Method: in how many Lights he placeth his Fluxions: and in what different ways he attempts to demonstrate the same Point: one would be inclined to think, he was himself suspicious of the justness of his own demonstrations; and that he was not enough pleased with any one notion steadily to adhere to it. Thus much at least is plain, that he owned himself satisfied concerning certain Points, which nevertheless he could not undertake to demonstrate to others. [NOTE: See Letter to Collins, Nov. 8, 1676.] Whether this satisfaction arose from tentative Methods or Inductions; which have often been admitted by Mathematicians (for instance by Dr. Wallis in his Arithmetic of Infinites) is what I shall not pretend to determine. But, whatever the Case might have been with respect to the Author, it appears that his Followers have shewn themselves more eager in applying his Method, than accurate in examining his Principles.

XVIII. It is curious to observe, what subtilty and skill this great Genius employs to struggle with an insuperable Difficulty; and through what Labyrinths he endeavours to escape the Doctrine of Infinitesimals; which as it intrudes upon him whether he will or no, so it is admitted and embraced by others without the least repugnance. Leibnitz and his followers in their calculus differentialis making no manner of scruple, first to suppose, and secondly to reject Quantities infinitely small: with what clearness in the Apprehension and justness in the reasoning, any thinking Man, who is not prejudiced in favour of those things, may easily discern. The Notion or Idea of an infinitesimal Quantity, as it is an Object simply apprehended by the Mind, hath been already considered. [NOTE: Sect. 5 and 6.] I shall now only observe as to the method of getting rid of such Quantities, that it is done without the least Ceremony. As in Fluxions the Point of first importance, and which paves the way to the rest, is to find the Fluxion of a Product of two indeterminate Quantities, so in the calculus differentialis (which Method is supposed to have been borrowed from the former with some small Alterations) the main Point is to obtain the difference of such Product. Now the Rule for this is got by rejecting the Product or Rectangle of the Differences. And in general it is supposed, that no Quantity is bigger or lesser for the Addition or Subduction of its Infinitesimal: and that consequently no error can arise from such rejection of Infinitesimals.
XIX. And yet it should seem that, whatever errors are admitted in the Premises, proportional errors ought to be apprehended in the Conclusion, be they finite or infinitesimal: and that therefore the of Geometry requires nothing should be neglected or rejected. In answer to this you will perhaps say, that the Conclusions are accurately true, and that therefore the Principles and Methods from whence they are derived must be so too. But this inverted way of demonstrating your Principles by your Conclusions, as it would be peculiar to you Gentlemen, so it is contrary to the Rules of Logic. The truth of the Conclusion will not prove either the Form or the Matter of a Syllogism to be true: inasmuch as the Illation might have been wrong or the Premises false, and the Conclusion nevertheless true, though not in virtue of such Illation or of such Premises. I say that in every other Science Men prove their Conclusions by their Principles, and not their Principles by the Conclusions. But if in yours you should allow yourselves this unnatural way of proceeding, the Consequence would be that you must take up with Induction, and bid adieu to Demonstration. And if you submit to this, your Authority will no longer lead the way in Points of Reason and Science.

XX. I have no Controversy about your Conclusions, but only about your Logic and Method. How you demonstrate? What Objects you are conversant with, and whether you conceive them clearly? What Principles you proceed upon; how sound they may be; and how you apply them? It must be remembered that I am not concerned about the truth of your Theorems, but only about the way of coming at them; whether it be legitimate or illegitimate, clear or obscure, scientific or tentative. To prevent all possibility of your mistaking me, I beg leave to repeat and insist, that I consider the Geometrical Analyst as a Logician, i.e. so far forth as he reasons and argues; and his Mathematical Conclusions, not in themselves, but in their Premises; not as true or false, useful or insignificant, but as derived from such Principles, and by such Inferences. And forasmuch as it may perhaps seem an unaccountable Paradox, that Mathematicians should deduce true Propositions from false Principles, be right in the Conclusion, and yet err in the Premises; I shall endeavour particularly to explain why this may come to pass, and shew how Error may bring forth Truth, though it cannot bring forth Science.

XXI. In order therefore to clear up this Point, we will suppose for instance that a Tangent is to be drawn to a Parabola, and examine the progress of this Affair, as it is performed by infinitesimal Differences.

Let \( AB \) be a Curve, the Abscisse \( AP = x \), the Ordinate \( PB = y \), the Difference of the Abscisse \( PM = dx \), the Difference of the Ordinate \( RN = dy \). Now by supposing the Curve to be a Polygon, and consequently \( BN \), the Increment or Difference of the Curve, to be a straight Line coincident with the Tangent, and the differential Triangle \( BRN \) to be similar to the triangle \( TPB \) the Subtangent \( PT \) is found a fourth Proportional to \( RN : RB : PB : \) that is to \( dy : dx : y \). Hence the Subtangent will be

But herein there is an error arising from the aforementioned false supposition, whence the value of \( PT \) comes out greater than the Truth: for in reality
it is not the Triangle $RNB$ but $RLB$ which is similar to $PBT$, and therefore (instead of $RN$) $RL$ should have been the first term of the Proportion, i. e. $RN + NL$, i. e. $dy + z$: whence the true expression for the Subtangent should have been.

There was therefore an error of defect in making $dy$ the divisor: which error was equal to $z$, i. e. $NL$ the Line comprehended between the Curve and the Tangent. Now by the nature of the Curve $yy = px$, supposing $p$ to be the Parameter, whence by the rule of Differences $2ydy = pdx$ and

But if you multiply $y + dy$ by itself, and retain the whole Product without rejecting the Square of the Difference, it will then come out, by substituting the augmented Quantities in the Equation of the Curve, that

truly. There was therefore an error of excess in making

which followed from the erroneous Rule of Differences. And the measure of this second error is

Therefore the two errors being equal and contrary destroy each other; the first error of defect being corrected by a second error of excess.

XXII. If you had committed only one error, you would not have come at a true Solution of the Problem. But by virtue of a twofold mistake you arrive, though not at Science, yet at Truth. For Science it cannot be called, when you proceed blindfold, and arrive at the Truth not knowing how or by what means. To demonstrate that $z$ is equal to

let $BR$ or $dx$ be $m$ and $RN$ or $dy$ be $n$. By the thirty third Proposition of the first Book of the Conics of Apollonius, and from similar Triangles, as $2x$ to $y$ so is $m$ to

Likewise from the Nature of the Parabola $yy + 2yn + nn = xp + mp$, and $2yn + nn = mp$: wherefore

and because $yy = px$,

will be equal to $x$. Therefore substituting these values instead of $m$ and $x$ we shall have

i. e.

which being reduced gives

XXIII. Now I observe in the first place, that the Conclusion comes out right, not because the rejected Square of $dy$ was infinitely small; but because this error was compensated by another contrary and equal error. I observe in the second place, that whatever is rejected, be it every so small, if it be real, and consequently makes a real error in the Premises, it will produce a proportional real error in the Conclusion. Your Theorems therefore cannot be accurately true, nor your Problems accurately solved, in virtue of Premises, which themselves are not accurate, it being a rule in Logic that Conclusio sequitur partem debiliorem. Therefore I observe in the third place, that when the Conclusion is evident and the Premises obscure, or the Conclusion accurate and the Premises inaccurate, we may safely pronounce that such Conclusion is neither evident nor accurate, in virtue of those obscure inaccurate Premises or Principles; but in virtue of
some other Principles which perhaps the Demonstrator himself never knew or thought of. I observe in the last place, that in case the Differences are supposed finite Quantities ever so great, the Conclusion will nevertheless come out the same: inasmuch as the rejected Quantities are legitimately thrown out, not for their smallness, but for another reason, to wit, because of contrary errors, which destroying each other do upon the whole cause that nothing is really, though something is apparently thrown out. And this Reason holds equally, with respect to Quantities finite as well as infinitesimal, great as well as small, a Foot or a Yard long as well as the minutest Increment.

XXXI. A Point may be the limit of a Line: A Line may be the limit of a Surface: A Moment may terminate Time. But how can we conceive a Velocity by the help of such Limits? It necessarily implies both Time and Space, and cannot be conceived without them. And if the Velocities of nascent and evanescent Quantities, i. e. abstracted from Time and Space, may not be comprehended, how can we comprehend and demonstrate their Proportions? Or consider their rationes primae and ultimae? For to consider the Proportion or Ratio of Things implies that such Things have Magnitude: That such their Magnitudes may be measured, and their Relations to each other known. But, as there is no measure of Velocity except Time and Space, the Proportion of Velocities being only compounded of the direct Proportion of the Spaces, and the reciprocal Proportion of the Times; doth it not follow that to talk of investigating, obtaining, and considering the Proportions of Velocities, exclusively of Time and Space, is to talk unintelligibly?

XXXV. I know not whether it be worth while to observe, that possibly some Men may hope to operate by Symbols and Suppositions, in such sort as to avoid the use of Fluxions, Momentums, and Infinitesimals after the following manner. Suppose x to be one Absciss of a Curve, and z another Absciss of the same Curve. Suppose also that the respective Areas are $xxx$ and $zzz$; and that $z - x$ is the Increment of the Absciss, and $zzz - xxx$ the Increment of the Area, without considering how great, or how small those Increments may be. Divide now $zzz - xxx$ by $z - x$ and the Quotient will be $zz + zx + xx$: and, supposing that $z$ and $x$ are equal, this same Quotient will be $3xx$ which in that case is the Ordinate, which therefore may be thus obtained independently of Fluxions and Infinitesimals. But herein is a direct Fallacy: for in the first place, it is supposed that the Abscisses $z$ and $x$ are unequal, without such supposition no one step could have been made; and in the second place, it is supposed they are equal: which is a manifest Inconsistency, and amounts to the same thing that hath been before considered. [NOTE: Sect. 15.] And there is indeed reason to apprehend, that all Attempts for setting the abstruse and fine Geometry on a right Foundation, and avoiding the Doctrine of Velocities, Momentums, &c. will be found impracticable, till such time as the Object and the End of Geometry are better understood, than hitherto they seem to have been. The great Author of the Method of Fluxions felt this Difficulty, and therefore he gave in to those nice Abstractions and Geometrical Metaphysics, without which he saw nothing could be done on the received Principles; and what in the way of
Demonstration he hath done with them the Reader will judge. It must, indeed, be acknowledged, that he used Fluxions, like the Scaffold of a building, as things to be laid aside or got rid of, as soon as finite Lines were found proportional to them. But then these finite Exponents are found by the help of Fluxions. Whatever therefore is got by such Exponents and Proportions is to be ascribed to Fluxions: which must therefore be previously understood. And what are these Fluxions? The Velocities of evanescent Increments? And what are these same evanescent Increments? They are neither finite Quantities nor Quantities infinitely small, nor yet nothing. May we not call them the Ghosts of departed Quantities?

8.2 MacLaurin’s response. 1742.

Maclaurin’s treatment of fluxions. Need picture.

Maclaurin

A treatise of fluxions. 1742. II.496.

496. In general suppose, as in art. 66, that while the point P (fig. 220) describes the right line Aa with an uniform motion, the point M sets out from L with a velocity that is to the constant velocity of P as Lc to Dg, and proceeds in the right line Ee with a motion continually accelerated or retarded, that LS any space described by M is always to DG the space described in the same time by P as Lf to Dg, that cx is to Dg as the difference of the velocities of M at S and L to the constant velocity of P, and that LS is always to LC as Lf to Lc. Then LS being always expressed by LC + CS, it is manifest that (since LC is to DG as Lc to Dg, or as the velocity of M at L to the velocity of P) LC is what would have been described by M if its motion had continued uniformly from L, and that CS arises in this expression in consequence of the acceleration or retardation of the motion of the point M while it describes LS. But if LS and DG be supposed indefinitely small increments of EL and AD, cx will be infinitely less than Dg; and since cf is less than cx by what was shown in art. 66, it follows that cf will be infinitely less than Lc, and CS infinitely less than LC. Therefore when the increment LS is supposed indefinitely small, and its expression is resolved into two parts LC and CS, of which the former LC is always in the same ratio to DG (the simultaneous increment of AD while the increments vary, and the latter CS is infinitely less than the former LC, we may conclude that the part CS is that which arises in consequence of the variation of the motion of M while it describes LS, and is therefore to be neglected in measuring the motion of M at L, or the fluxion of the right line EL. Thus the manner of investigating the differences or fluxions of quantities in the method of infinitesimals maybe deduced from the principles of the method of fluxions demonstrated above. For instead of neglecting CS because it is infinitely less than LC (according to the usual manner of reasoning in that method), we may reject it, because we may hence conclude that it is not produced in consequence
of the generating motion at $L$, but of the subsequent variations of this motion. And it appears why the conclusions in the method of infinitesimals are not to be represented as if they were only near the truth, but are to be held as accurately true.

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8.3 45b. Maclaurians definition of limit

Definition of Limits

Maclaurin
A treatise of fluxions
1742, sections 502-503.

502. But however safe and convenient this method may be, some will always scruple to admit infinitely little quantities, and infinite orders of infinitesimals, into a science that boasts of the most evident and accurate principles as well as of the most rigid demonstrations; and therefore we chose to establish so extensive and useful a doctrine in the preceding chapters on more unexceptionable postulata. In order to avoid such suppositions, Sir Isaac Newton considers the simultaneous increments of the flowing quantities as finite, and then investigates the ratio which is the limit of the various proportions which those increments bear to each other, while he supposes them to decrease together till they vanish; which ratio is the same with the ratio of the fluxions by what was shown in art. 66, 67, and 68. In order to discover this limit, he first determines the ratio of the increments in general, and reduces it to the most simple terms, so as that (generally speaking) a part at least of each term may be independent of the value value of the increments themselves; then by supposing the increments to decrease till they vanish, the limit readily appears.

503. For example, let $a$ be an invariable quantity, $x$ a flowing quantity, and $o$ any increment of $x$; then the simultaneous increments of $xx$ and $ax$ will be $2'xo + oo$ and $ao$, which are in the same ratio to each other as $2x + o$ is to $a$. This ratio of $2x + o$ to $a$ continually decreases while $o$ decreases, and is always greater than the ratio of $2x$ to $a$ while $o$ is any real increment, but it is manifest that it continually approaches to the ratio of $2x$ to $a$ as its limit; whence it follows that the fluxion of $xx$ is to the fluxion of $ax$ as $2x$ is to $a$. If $x$ be supposed to flow uniformly, $ax$ will likewise flow uniformly, but $xx$ with a motion continually accelerated: the motion with which $ax$ flows may be measured by $ao$, but the motion with which $xx$ flows is not to be measured by its increment $2xo + oo$ (by ax. 1), but by the part $2xo$ only, which is generated in consequence of that motion; and the part $oo$ is to be rejected because it is generated in consequence only of the acceleration of the motion with which the variable square flows, while $o$ the increment of its side is generated: and the ratio of $2xo$ to $ao$ is that of $2x$ to $a$, which was found to be the limit of the ratio of the increments $2xo + oo$ and $ao$ (fig. 220).
8.4 Euler’s definition of continuity. 1748.

[ref 46]

8.5 d’Alembert’s response to Berkeley. 1754.

[ref 47]

8.6 Euler, infinitely small. 1755.

[ref 48]

82. But to return to the problem, although some may deny that an infinite number truly exists in the world, nevertheless in mathematical speculations there very often occur questions, to which, unless an infinite number is admitted, it is not possible to respond. Thus, if there is sought the sum of all the numbers that constitute this series 1 + 2 + 3 + 4 + 5+ etc., since these numbers progress without end, and increase, the sum of all of them could certainly not be finite; and therefore it comes about that it must be infinite. To denote a quantity of this kind mathematicians us this sign $\infty$, by which is meant a quantity greater than any finite, or assignable, quantity. Thus when a parabola is defined in such a way that it is said to be an infinitely long ellipse, we could correctly assert that the axis of the parabola is an infinitely long straight line.

83. Moreover, this teaching on the infinite will be better illustrated, if we explain what would mathematically be infinitely small. There is no doubt that every quantity can be decreased so far that it simply vanishes, and goes to nothing. But an infinitely small quantity is nothing other than a vanishing quantity, and is therefore really equal to 0. This definition of the infinitely small also fits those quantities that are said to be less than every assignable quantity; for if a quantity becomes so small, that is less than any assignable quantity, it certainly cannot be other than nothing; for unless it were 0, it would be possible to assign a quantity equal to it, which is against the hypothesis. Therefore, to those asking what an infinitely small quantity is in mathematics, we reply that it is really 0; therefore, there are not so many mysteries hidden in this as are commonly thought, and which to many render the calculus of the infinitely small in some way suspect. Meanwhile, if they ignore them in what follows, where we teach this calculus, their doubts will be completely removed.

84. Therefore since we have shown an infinitely small quantity to be really zero, the first objection is: why do we not always denote infinitely small quantities by the sign 0, but employ special signs to denote them? For since all nothings are equal amongst themselves, it seems superfluous to denote them by different signs. But although any two zeros are equal amongst themselves, so that their difference becomes nothing, nevertheless, when the two are compared in some way, either arithmetically or geometrically, there we see a difference
between them, in fact the quantity arising from the comparison; indeed the
arithmetical ratio between two such zeros is equality, but not the geometric ra-
tio. This is very easily seen from the geometric proportion $2 : 1 = 0 : 0$, in which
the fourth term is 0, as is the third. Moreover from the nature of proportions,
since the first term is twice as great as the second, it must also be that the third
is twice as great as the fourth.

85. Moreover this is also very plain in common arithmetic: for it is known
to everyone that zero multiplied by any number gives zero, and that $n \cdot 0 = 0$,
and thus it will be that $n : 1 = 0 : 0$. Whence it is clear that it is possible for
two zeros to have any geometric proportion between them, although looking at
the thing arithmetically, their ratio is always one of equality. Therefore since it
is possible to introduce any ratio between zeros, different symbols are used to
indicate this diversity, since usually it is the theometric ratio between different
zeros that is to be investigated. Indeed, in the calculus of the infinitely small
we are concerned with nothing other than geometric ratios between different
infinitely small quantities, and in this matter, therefore, unless we use different
signs to indicate them, we will be led into great confusion, and with no way to
escape from it.

86. If, therefore, as is the accepted way of writing in the analysis of infinities,
$dx$ denotes an infinitely small quantity, we will have both $dx = 0$ and $a \cdot dx = 0$, where $a$
denotes any finite quantity. This is so notwithstanding that the
geometric ratio $a \cdot dx : dx$ is finite, namely as $a : 1$; and here, on account of
that, these two infinitely small quantities $dx$ and $a \cdot dx$, although both are equal
to 0, should not be confused with each other, if indeed their ratio is to be
investigated. Similarly, if there occur different infinitely small quantities $dx$ and
dy, although both are equal to 0, nevertheless their ratio is not constant. And
to the investigation of ratios between two infinitely small quantities of this kind,
there is applied all the power of the differential calculus. Moreover the use of
such comparison, although it may seem at first sight very limited, is nevertheless
found to be exceedingly broad, as from here will appear much more closely.

8.7 d'Alambert, encyclopedia

Leibniz was embarrassed by the objections he felt to exist against infinitely small
quantities, as the appear

in the differential calculus; thus he preferred to reduce infinitely small to
merely incomparable quantities.

... Newton started out from another principle; and one can say that the meta-
physics of this great mathematician on the calculus of fluxions is very exact
and illuminating, even though he allowed us only an imperfect glimpse of his
thoughts. He never considered the differential calculus as the study of infinitely
small quantities, but as the method of first and ultimate ratios, that is to say,
the method of finding the limits of ratios. Thus this famous author has never
differentiated quantities but only equations; in fact, every equation involves a relation between two variables and the differentiation of equations merely consists in finding the limit of the ratio of the finite differences of the two quantities contained in the equation.

... Once this is well understood, one will feel that the assumption made concerning infinitely small quantities serves only to abbreviate and simplify the reasoning; but that the differential calculus does not necessarily suppose the existence of those quantities; and that moreover this calculus merely consists in algebraically determining the limit of a ratio, for which we already have the expression in terms of lines, and in equating those two expressions. This will provide us with one of the lines we are looking for. This is perhaps the most precise and neatest possible definition of the differential calculus; but it can be understood only when one is well acquainted with this calculus, because often the true nature of a science can understood only by those that have studies this science.

... We have seen above that in the differential calculus there are really no infinitely small quantities of the first order; that actually those quantities called \( u \) are supposed to be divided by other supposedly infinitely small quantities; in this state they do not denote either infinitely small quantities or quotients of infinitely small quantities; they are the limits of the ratio of two finite quantities. This same holds for the second-order differences and for those of higher order. There is actually no quantity in Geometry such as \( ddy \); whenever \( ddy \) occurs in an equation it is supposed to be divided by a quantity \( dx^2 \), or another of the same order. What now is \( ddy/dx^2 \)? It is the limit of the ratio \( ddy/dx \) divided by \( dx \); or, what is still clearer, it is the limit of \( dz/dx \), where \( dy/dx = z \) is a finite quantity.

from kant to hilbert, a source book in the foundation of mathematics, WB Ewald, pages 126-128.

8.8 d’Alembert definition of a limit. 1765.

d’Alembert

from d’Alembert and Diderot Encyclopédie, 1751–65, IX, 542.
[49]

LIMIT (Mathematics). One says that a magnitude is the limit of another magnitude, when the second may approach the first more closely than by a given quantity, as small as one wishes, moreover without the magnitude which approaches being allowed ever to surpass the magnitude that it approaches; so that the difference between such a quantity and its limit is absolutely unassignable.

For example, suppose we have two polygons, one inscribed in a circle and
the other circumscribed; it is clear that one may increase the number of sides as much as one wishes, and in that case each polygon will approach ever more closely to the circumference of the circle; the perimeter of the inscribed polygon will increase and that of the circumscribed polygon will decrease, but the perimeter or edge of the first will never surpass the length of the circumference, and that of the second will never be smaller than that same circumference; the circumference of the circle is therefore the \textit{limit} of the increase of the first polygon and of the decrease of the second.

1. If two magnitudes are the \textit{limit} of the same quantity, the two magnitudes will be equal to each other.

2. Suppose \( A \times B \) is the product of two magnitudes \( A, B \). Let us suppose that \( C \) is the \textit{limit} of the magnitude \( A \), and \( D \) the \textit{limit} of the quantity \( B \); I say that \( C \times D \), the product of the \textit{limits}, will necessarily be the \textit{limit} of \( A \times B \), the product of the magnitude \( A, B \).

These two propositions, which one will find demonstrated exactly in the \textit{Institutions de Géométrie}, serve as principles for demonstrating rigorously that one has the area of a circle from multiplying its semicircumference by its radius. See the work cite, p. 331 and following in the second volume.

The theory of \textit{limits} is the foundation of the true justification of the differential calculus. \textit{SEE DIFFERENTIAL, FLUXION, EXHAUSTION, INFINITE.}

Strictly speaking, the \textit{limit} never coincides, or never becomes equal to the quantity of which it is the \textit{limit}, but the latter approaches it ever more closely, and may differ from it as little as one wishes. The circle, for example, is the \textit{limit} of the inscribed and circumscribed polygons; for strictly it never coincides with them, although they may approach it indefinitely. This notion may serve to clarify several mathematical propositions. For example, one says that the sum of a decreasing geometric progression in which the first term is \( a \) and the second is \( b \), is \( \frac{an}{a-b} \); this value is never strictly the sum of the progression, it is the \textit{limit} of that sum, that is to say, the quantity which it may approach as closely as one wishes, without ever arriving at it exactly. For if \( e \) is the last term in the progression, the exact value of the sum is \( \frac{ae}{a-b} \), which is always less than \( \frac{an}{a-b} \) because even in a decreasing geometric progression, the last term \( e \) is never 0; but as this term continually approaches zero, without ever arriving at it, it is clear that zero is its \textit{limit}, and that consequently the \textit{limit} of \( \frac{ae}{a-b} \) is \( \frac{an}{a-b} \), supposing \( e = 0 \), that is to say, on putting in place of \( e \) its \textit{limit}.

[ref 49]

8.9 Euler’s introduction to integration. 1768

[ref 50]

[[useful for its terminology, its relating integral and diff'l calculus, its discussion of infinitesimals]]

Foundations of Integral Calculus
1768.
(Defn’s 1, 2 from third edition, 1824)

PRELIMINARY NOTES
ON INTEGRAL CALCULUS
IN GENERAL

Definition 1

1. Integral calculus is the method of finding, from a given relationship between differentials, a relationship between the quantities themselves: and the operation by which this is carried out is usually called integration.

Corollary 1

2. Therefore where differential calculus teaches us to investigate the relationship between differentials from a given relationship between variable quantities, integral calculus supplies us with the inverse method.

Corollary 2

3. Clearly just as in Analysis two operations are always contrary to each other, as subtraction to addition, division to multiplication, extraction of roots to raising of power, so also by similar reasoning integral calculus is contrary to differential calculus.

Corollary 3

4. Given any relationship between two variable quantities \(x\) and \(y\), in differential calculus there is taught a method of investigating the ratio of the differentials \(dx : dy\); but if from this ratio of differentials there can in turn be determined the relationship of the quantities \(x\) and \(y\), this matter is assigned to integral calculus.\(^1\)

Commentary 1

5. I have already noted that in differential calculus the question of differentials must be understood not absolutely but relatively, thus if \(y\) is any function of \(x\), it is not the differential \(dy\) itself but the ratio to the differential \(dx\) that is determined. For since all differentials in themselves are equal to nothing, whatever may be the function of \(y\) of \(x\), always \(dy = 0\), and thus it is not possible to search more generally for anything absolute. But the question must be rightly proposed thus, that while \(x\) takes and infinitely small and therefore vanishing

\(^1\)Euler used the notation \(\partial x : \partial y\) where we now use \(dx : dy\). His notation has been replaced by the modern throughout. JS
increment $dx$, there is defined a ratio of the increment of the function $y$, which it takes as a result, to $dx$; for although both are $= 0$, nevertheless there stands a definite ratio between them, which is correctly investigated by differential calculus. Thus if $y = xx$, it is shown by differential calculus that $\frac{dy}{dx} = 2x$, nor is this ratio of increments true unless the increment $dx$, from which $dy$ arises, is put equal to nothing. But nevertheless, having observed this truth about differentials, one can tolerate common language, in which differentials are spoken of as absolutes, which always at least in the mind referring to the truth. Properly, therefore, we say if $y = xx$ then $dy = 2x \, dx$, even though it would not be false if anyone said $dy = 3x \, dx$ or $dy = 4x \, dx$, for since $dx = 0$ and $dy = 0$, these equalities would equally well stand; but only the first of the ratios, $\frac{dy}{dx} = 2x$, is agreed to be true.

Commentary 2

6. In the same way that the differential calculus is called by the English the method of fluxions, so integral calculus is usually called by them the inverse method of fluxions, since indeed one reverts from fluxions to fluent quantities. For what we call variable quantities, the English more fitly call by the name of fluent quantities, and their infinitely small or vanishing increments they call fluxions, so that fluxions are the same to them as differentials to us. This variation in language is already established in use, so that a reconciliation is scarcely ever to be expected; indeed we imitate the English freely in forms of speech, but the notation that we use seems to have been established a long time before their notation. And indeed since so many books are already published written either way, a reconciliation of this kind would be of no use.

Definition 2

7. Since the differentiation of any function of $x$ has a form of this kind $X \, dx$, when such a differential form $X \, dx$ is proposed, in which $X$ is any function of $x$, that function whose differential $= X \, dx$ is called its integral, and is usually indicated by the prefix $\int$, so that $\int X \, dx$ denotes that variable quantity whose differential $= X \, dx$.

Corollary 1

8. Therefore from the integral of the proposed differential $X \, dx$, or from the function of $x$ whose differential $= X \, dx$, both of which will be indicated by this notation $\int X \, dx$, there is to be investigated whatever is to be explained by integral calculus.

Corollary 2

9. Therefore just as the letter $d$ is the sign of differentiation, so we use the letter $\int$ as the sign of integration, and thus these two signs are mutually
contrary to each other, as though they destroy each other: certainly $\int dX = X$, because the former is denoted by the quantity whose differential is $dX$, which in both cases is $X$.

Corollary 3

10. Therefore since the differentials of these functions of $x$

$$x^2, x^n, \sqrt{aa - xx}$$

are

$$2x \, dx, nx^{n-1} \, dx, \frac{-x \, dx}{\sqrt{(aa - xx)}}$$

then adjoining the sign of integration $\int$, they are seen to become:

$$\int 2x \, dx = xx; \int nx^{n-1} \, dx = x^n; \int \frac{-x \, dx}{\sqrt{(aa - xx)}} = \sqrt{(aa - xx)}$$

whence the use of this sign is more clearly seen.

8.10 Lagrange ‘avoids’ infinitesimals, on arbitrariness small intervals. 1797.

[ref 51, 51b]

8.11 Bolzano, greatest lower bound, continuity. 1817.

[[shows how complicated ‘continuous’ is]]

Bernard Bolzano

“Purely analytic proof of the theorem that between any two values which give results of opposite sign, there lies at least one real root of the equation.”

1817. [Ste 53]


I. The most common kind of proof [of the intermediate value theorem\footnote{In its simplest form, the Intermediate Value Theorem states that if $f(x)$ is a continuous function on the interval $a \leq x \leq b$ and if $f(a) < 0$ and $f(b) > 0$, there is some value $c$, $a < c < b$ with the property that $f(c) = 0.$}] depends on a truth borrowed from geometry, namely: that every continuous line of simple curvature of which the ordinates are first positive and then negative (or conversely) must necessarily intersect the abscissa–line somewhere at a point lying between these ordinates. There is certainly nothing to be said against
the correctness, nor against the obviousness of this geometrical proposition. But it is also equally clear that it is an unacceptable breach of it good method to try to derive truths of pure (or general) matheamtics (i.e., arithmetic, algebra, analysis) from considerations which belong to merely applied (or special) part of it, namely geometry. Indeed, have we not long felt, and acknowledged, the impropriety of such a crossing to another kind? Are there not a hundred other cases where a method of avoiding this has been discovered, and where the avoidance was considered a virtue? So if we were to be consistent must we not strive to do the same here? In fact, anyone who considers that scientific proofs should not be merely confirmations, but rather groundings, i.e., presentations of the objective reason of the truth to be proved, realizes at once that the strictly scientific proof, or the objective reason of a truth, which hold equally for all quantities, whether in space or not, cannot possibly lie in a truth which hold merely for quantities which are in space. If we adhere to this view we see instead that such a geometrical proof is, in this as in most cases, really circular. For which the geometrical truth to which we refer here is (as we have already admitted) extremely obvious and therefore needs no proof in the sense of confirmation, it none the less does need a grounding. For it is apparent that the concepts of which it consists are so combined that we cannot hesitate for a moment to say that it cannot possibly be one of those simple truths, which are called axioms, or basic truths, because they are the basis for other truths and are not themselves consequences. On the contrary, it is a theorem of consequent truth, i.e., a kind of truth that has its basis in certain other truths and therefore, in science, must be proved by a derivation from these other truths. Now consider, if you will, the objective reason why a line, as described above, intersects its abscissae–line. Surely everyone will soon see that this reason lies in nothing other than that general truth, as a result of which every continuous function of $x$ which is positive for one value of $x$, and negative for another, must be zero for some intermediate value of $x$. And this is precisely the truth which is to be proved here. It is therefore quite wrong to have allowed the latter to be derived from the former (as happens in the kind of proof we are now examining). Rather, conversely, the former must be derived from the latter if we intend to represent the truths in science exactly as they are related to each other in their objective connection.

II. The proof which some people have produced from the concept of the continuity of a function mixed up with the concepts of time and motion, is no less objectionable. ‘If two function $fx$ and $\phi x$', they say, ‘vary according to the law of continuity, and if for $x = \alpha$, $f \alpha < \phi \alpha$, but for $x = \beta$, $f \beta > \phi \beta$, then there must be some value $u$, lying between $\alpha$ and $\beta$, for which $fu = \phi u$. For if we imagin that the variable quantity $x$ in both these functions successively takes all values between $\alpha$ and $\beta$, and in both always takes the same value at the same moment, then at the beginning of this continuous change in the value of $x$, $fx < \phi x$, and at the end, $fx > \phi x$. But since both functions, by virtue of their continuity, must first go through all intermediate values before they can reach a higher value, there must be some intermediate moment at which they
were both equal to one another.' This is further illustrated by the example of
the motion of two bodies, of which one is initially behind the other and later ahead of the other. It necessarily follows that at one time it must have passed the other.

No one will want to deny that the concept of time, as well as that of motion, is just as alien to general mathematics as the concept of space. Nevertheless we would have no objection if these two concepts were only introduced here for the sake of clarification. For we are in no way party to a purism so exaggerated, that it demands, in order to keep the science free from everything alien, that in its exposition one cannot even use an expression borrowed from another field, even if only in a figurative sense and with the purpose of describing a fact more briefly and clearly than could be done in a description involving purely specialist terms. Nor do we object to such use if it is just to avoid the monotony of constant repetition of the same word, or to remind us, by the mere name given to a thing, of an example which could serve to confirm a claim. It follows immediately that we do not regard examples and applications as detracting in the least from the perfection of a scientific exposition. There is only one thing that we do strictly require: that examples never be put forward instead of proofs, and that the essence of a deduction never be based on the merely figurative use of phrases or on associated ideas, so that the deduction itself become void as soon as these are changed.

In accordance with these views, the inclusion of the concept of time in the above proof may still perhaps be excused, because no conclusion is based on phrases containing it, which would not also hold without it. But the last illustration using the motion of a body cannot be regarded as anything more than a mere example which does not prove the proposition itself, but instead must first be proved by it.
Chapter 9

Foundations and the ’Modern’ calculus

9.1 Berkeley’s objections. 1734

[ref 44] http://ia361301.us.archive.org/7/items/theanalystoradis00berkuoft/theanalystoradis00berkuoft.pdf

THE ANALYST;
OR, A
DISCOURSE
Addressed to an
Infidel MATHEMATICIAN.
WHEREIN
It is examined whether the Object, Principles, and Inferences of the modern Analysis are more distinctly conceived, or more evidently deduced, than Religious Mysteries and Points of Faith.

by the author of
THE MINUTE PHILOSPHER

’First cast out the beam out of thine own eye; and then shalt thou see clearly to cast out the mote out of thy brother’s eye.’
MATT. c. vii. v 5

First published in 1734

I. Though I am a Stranger to your Person, yet I am not, Sir, a Stranger to the Reputation you have acquired, in that branch of Learning which hath
been your peculiar Study; nor to the Authority that you therefore assume in things foreign to your Profession, nor to the Abuse that you, and too many more of the like Character, are known to make of such undue Authority, to the misleading of unwary Persons in matters of the highest Concernment, and whereof your mathematical Knowledge can by no means qualify you to be a competent Judge. Equity indeed and good Sense would incline one to disregard the Judgment of Men, in Points which they have not considered or examined. But several who make the loudest Claim to those Qualities, do, nevertheless, the very thing they would seem to despise, clothing themselves in the Livery of other Mens Opinions, and putting on a general deference for the Judgment of you, Gentlemen, who are presumed to be of all Men the greatest Masters of Reason, to be most conversant about distinct Ideas, and never to take things on trust, but always clearly to see your way, as Men whose constant Employment is the deducing Truth by the justest inference from the most evident Principles. With this bias on their Minds, they submit to your Decisions where you have no right to decide. And that this is one short way of making Infidels I am credibly informed.

II. Whereas then it is supposed, that you apprehend more distinctly, consider more closely, infer more justly, conclude more accurately than other Men, and that you are therefore less religious because more judicious, I shall claim the privilege of a Free-Thinker; and take the Liberty to inquire into the Object, Principles, and Method of Demonstration admitted by the Mathematicians of the present Age, with the same freedom that you presume to treat the Principles and Mysteries of Religion; to the end, that all Men may see what right you have to lead, or what Encouragement others have to follow you. It hath been an old remark that Geometry is an excellent Logic. And it must be owned, that when the Definitions are clear; when the Postulata cannot be refused, nor the Axioms denied; when from the distinct Contemplation and Comparison of Figures, their Properties are derived, by a perpetual well-connected chain of Consequences, the Objects being still kept in view, and the attention ever fixed upon them; there is acquired a habit of reasoning, close and exact and methodical: which habit strengthens and sharpens the Mind, and being transferred to other Subjects, is of general use in the inquiry after Truth. But how far this is the case of our Geometrical Analysts, it may be worth while to consider.

III. The Method of Fluxions is the general Key, by help whereof the modern Mathematicians unlock the secrets of Geometry, and consequently of Nature. And as it is that which hath enabled them so remarkably to outgo the Ancients in discovering Theorems and solving Problems, the exercise and application thereof is become the main, if not sole, employment of all those who in this Age pass for profound Geometers. But whether this Method be clear or obscure, consistent or repugnant, demonstrative or precarious, as I shall inquire with the utmost impartiality, so I submit my inquiry to your own Judgment, and that of every candid Reader. Lines are supposed to be generated [NOTE: Introd. ad Quadraturam Curvarum.] by the motion of Points, Planes by the motion of Lines, and Solids by the motion of Planes. And whereas Quantities generated
9.1. BERKELEY’S OBJECTIONS. 1734

in equal times are greater or lesser, according to the greater or lesser Velocity, wherewith they increase and are generated, a Method hath been found to determine Quantities from the Velocities of their generating Motions. And such Velocities are called Fluxions: and the Quantities generated are called flowing Quantities. These Fluxions are said to be nearly as the Increments of the flowing Quantities, generated in the least equal Particles of time; and to be accurately in the first Proportion of the nascent, or in the last of the evanescent, Increments. Sometimes, instead of Velocities, the momentaneous Increments or Decrements of undetermined flowing Quantities are considered, under the Appellation of Moments.

IV. By Moments we are not to understand finite Particles. These are said not to be Moments, but Quantities generated from Moments, which last are only the nascent Principles of finite Quantities. It is said, that the minutest Errors are not to be neglected in Mathematics: that the Fluxions are Celerities, not proportional to the finite Increments though ever so small; but only to the Moments or nascent Increments, whereof the Proportion alone, and not the Magnitude, is considered. And of the aforesaid Fluxions there be other Fluxions, which Fluxions of Fluxions are called second Fluxions. And the Fluxions of these second Fluxions are called third Fluxions: and so on, fourth, fifth, sixth, &c. ad infinitum. Now as our Sense is strained and puzzled with the perception of Objects extremely minute, even so the Imagination, which Faculty derives from Sense, is very much strained and puzzled to frame clear Ideas of the least Particles of time, or the least Increments generated therein: and much more so to comprehend the Moments, or those Increments of the flowing Quantities in statu nascenti, in their very first origin or beginning to exist, before they become finite Particles. And it seems still more difficult, to conceive the abstracted Velocities of such nascent imperfect Entities. But the Velocities of the Velocities, the second, third, fourth, and fifth Velocities, &c. exceed, if I mistake not, all Humane Understanding. The further the Mind analyseth and pursueth these fugitive Ideas, the more it is lost and bewildered; the Objects, at first fleeting and minute, soon vanishing out of sight. Certainly in any Sense a second or third Fluxion seems an obscure Mystery. The incipient Celerity of an incipient Celerity, the nascent Augment of a nascent Augment, i.e. of a thing which hath no Magnitude: Take it in which light you please, the clear Conception of it will, if I mistake not, be found impossible, whether it be so or no I appeal to the trial of every thinking Reader. And if a second Fluxion be inconceivable, what are we to think of third, fourth, fifth Fluxions, and so onward without end?

V. The foreign Mathematicians are supposed by some, even of our own, to proceed in a manner, less accurate perhaps and geometrical, yet more intelligible. Instead of flowing Quantities and their Fluxions, they consider the variable finite Quantities, as increasing or diminishing by the continual Addition or Subduction of infinitely small Quantities. Instead of the Velocities wherewith Increments are generated, they consider the Increments or Decrements themselves, which they call Differences, and which are supposed to be infinitely small. The Difference of a Line is an infinitely little Line; of a Plane an infinitely
little Plane. They suppose finite Quantities to consist of Parts infinitely little, and Curves to be Polygons, whereof the Sides are infinitely little, which by the Angles they make one with another determine the Curvity of the Line. Now to conceive a Quantity infinitely small, that is, infinitely less than any sensible or imaginable Quantity, or any the least finite Magnitude, is, I confess, above my Capacity. But to conceive a Part of such infinitely small Quantity, that shall be still infinitely less than it, and consequently though multiply’d infinitely shall never equal the minutest finite Quantity, is, I suspect, an infinite Difficulty to any Man whatsoever; and will be allowed such by those who candidly say what they think; provided they really think and reflect, and do not take things upon trust.

VI. And yet in the calculus differentialis, which Method serves to all the same Intents and Ends with that of Fluxions, our modern Analysts are not content to consider only the Differences of finite Quantities: they also consider the Differences of those Differences, and the Differences of the Differences of the first Differences. And so on ad infinitum. That is, they consider Quantities infinitely less than the least discernible Quantity; and others infinitely less than those infinitely small ones; and still others infinitely less than the preceding Infinitesimals, and so on without end or limit. Insomuch that we are to admit an infinite succession of Infinitesimals, each infinitely less than the foregoing, and infinitely greater than the following. As there are first, second, third, fourth, fifth &c. Fluxions, so there are Differences, first, second, third fourth, &c. in an infinite Progression towards nothing, which you still approach and never arrive at. And (which is most strange) although you should take a Million of Millions of these Infinitesimals, each whereof is supposed infinitely greater than some other real Magnitude, and add them to the least given Quantity, it shall be never the bigger. For this is one of the modest postulata of our modern Mathematicians, and is a Corner-stone or Ground-work of their Speculations.

VII. All these Points, I say, are supposed and believed by certain rigorous Exactors of Evidence in Religion, Men who pretend to believe no further than they can see. That Men, who have been conversant only about clear Points, should with difficulty admit obscure ones might not seem altogether unaccountable. But he who can digest a second or third Fluxion, a second or third Difference, need not, methinks, be squeamish about any Point in Divinity. There is a natural Presumption that Mens Faculties are made alike. It is on this Supposition that they attempt to argue and convince one another. What, therefore, shall appear evidently impossible and repugnant to one, may be presumed the same to another. But with what appearance of Reason shall any Man presume to say, that Mysteries may not be Objects of Faith, at the same time that he himself admits such obscure Mysteries to be the Object of Science?

VIII. It must indeed be acknowledged, the modern Mathematicians do not consider these Points as Mysteries, but as clearly conceived and mastered by their comprehensive Minds. They scruple not to say, that by the help of these new Analytics they can penetrate into Infinity it self: That they can even extend their Views beyond Infinity: that their Art comprehends not only Infinite,
but Infinite of Infinite (as they express it) or an Infinity of Infinites. But, notwithstanding all these Assertions and Pretensions, it may be justly questioned whether, as other Men in other Inquiries are often deceived by Words or Terms, so they likewise are not wonderfully deceived and deluded by their own peculiar Signs, Symbols, or Species. Nothing is easier than to devise Expressions or Notations for Fluxions and Infinitesimals of the first, second, third, fourth, and subsequent Orders, proceeding in the same regular form without end or limit . . . . &c. or \( dx \), \( d^2 dx \), \( d^3 dx \), \( d^4 dx \) &c. These Expressions indeed are clear and distinct, and the Mind finds no difficulty in conceiving them to be continued beyond any assignable Bounds. But if we remove the Veil and look underneath, if laying aside the Expressions we set ourselves attentively to consider the things themselves, which are supposed to be expressed or marked thereby, we shall discover much Emptiness, Darkness, and Confusion; nay, if I mistake not, direct Impossibilities and Contradictions. Whether this be the case or no, every thinking Reader is intreated to examine and judge for himself.

IX. Having considered the Object, I proceed to consider the Principles of this new Analysis by Momentums, Fluxions, or Infinitesimals; wherein if it shall appear that your capital Points, upon which the rest are supposed to depend, include Error and false Reasoning; it will then follow that you, who are at a loss to conduct your selves, cannot with any decency set up for guides to other Men. The main Point in the method of Fluxions is to obtain the Fluxion or Momentum of the Rectangle or Product of two indeterminate Quantities. Inasmuch as from thence are derived Rules for obtaining the Fluxions of all other Products and Powers; be the Coefficients or the Indexes what they will, integers or fractions, rational or surd. Now this fundamental Point one would think should be very clearly made out, considering how much is built upon it, and that its Influence extends throughout the whole Analysis. But let the Reader judge. This is given for Demonstration. [NOTE: Naturalis Philosophi principia mathematica, l. 2. lem. 2.] Suppose the Product or Rectangle \( AB \) increased by continual Motion: and that the momentaneous Increments of the Sides \( A \) and \( B \) are \( a \) and \( b \). When the Sides \( A \) and \( B \) were deficient, or lesser by one half of their Moments, the Rectangle was

\[
, i. e., .\]

And as soon as the Sides \( A \) and \( B \) are increased by the other two halves of their Moments, the Rectangle becomes

or . From the latter Rectangle subduct the former, and the remaining Difference will be \( aB + bA \). Therefore the Increment of the Rectangle generated by the entire Increments \( a \) and \( b \) is \( aB + bA \). Q.E.D. But it is plain that the direct and true Method to obtain the Moment or Increment of the Rectangle \( AB \), is to take the Sides as increased by their whole Increments, and so multiply them together, \( A + a \) by \( B + b \), the Product whereof \( AB + aB + bA + ab \) is the augmented Rectangle: whence if we subduct \( AB \), the Remainder \( aB + bA + ab \) will be the true Increment of the Rectangle, exceeding that which was obtained by the former illegitimate and indirect Method by the Quantity \( ab \). And this holds universally be the Quantities \( a \) and \( b \) what they will, big or little, Finite or Infinitesimal, Increments, Moments, or Velocities. Nor will it avail to say that
ab is a Quantity exceeding small: Since we are told that in rebus mathematicis
errores qum minimi non sunt contemnendi. [NOTE: Introd. ad Quadraturam Curvarum.]

X. Such reasoning as this for Demonstration, nothing but the obscurity of the
Subject could have encouraged or induced the great Author of the Fluxionary
Method to put upon his Followers, and nothing but an implicit deference to
Authority could move them to admit. The Case indeed is difficult. There can
be nothing done till you have got rid of the Quantity \( ab \). In order to this the
Notion of Fluxions is shifted: it is placed in various Lights: Points which should
be as clear as first Principles are puzzled; and Terms which should be steadily
used are ambiguous. But notwithstanding all this address and skill the point
of getting rid of \( ab \) cannot be obtained by legitimate reasoning. If a Man by
Methods, not geometrical or demonstrative, shall have satisfied himself of the
usefulness of certain Rules; which he afterwards shall propose to his Disciples
for undoubted Truths; which he undertakes to demonstrate in a subtile manner,
and by the help of nice and intricate Notions; it is not hard to conceive that
such his Disciples may, to save themselves the trouble of thinking, be inclined
to confound the usefulness of a Rule with the certainty of a Truth, and accept
the one for the other; especially if they are Men accustomed rather to compute
than to think; earnest rather to go on fast and far, than solicitous to set out
warily and see their way distinctly.

XI. The Points or meer Limits of nascent Lines are undoubtedly equal, as
having no more magnitude one than another, a Limit as such being no Quantity.
If by a Momentum you mean more than the very initial Limit, it must be either
a finite Quantity or an Infinitesimal. But all finite Quantities are expressly
excluded from the Notion of a Momentum. Therefore the Momentum must be an
Infinitesimal. And indeed, though much Artifice hath been employ’d to escape
or avoid the admission of Quantities infinitely small, yet it seems ineffectual. For
ought I see, you can admit no Quantity as a Medium between a finite Quantity
and nothing, without admitting Infinitesimals. An Increment generated in a
finite Particle of Time, is it self a finite Particle; and cannot therefore be a
Momentum. You must therefore take an Infinitesimal Part of Time wherein
to generate your Momentum. It is said, the Magnitude of Moments is not
considered: And yet these same Moments are supposed to be divided into Parts.
This is not easy to conceive, no more than it is why we should take Quantities
less than \( A \) and \( B \) in order to obtain the Increment of \( AB \), of which proceeding
it must be owned the final Cause or Motive is very obvious; but it is not so
obvious or easy to explain a just and legitimate Reason for it, or shew it to be
Geometrical.

XII. From the foregoing Principle so demonstrated, the general Rule for
finding the Fluxion of any Power of a flowing Quantity is derived. [NOTE:
Philosophi naturalis principia Mathematica, lib. 2. lem. 2.] But, as there seems
to have been some inward Scruple or Consciousness of defect in the foregoing
Demonstration, and as this finding the Fluxion of a given Power is a Point of pri-
mary Importance, it hath therefore been judged proper to demonstrate the same
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in a different manner independent of the foregoing Demonstration. But whether this other Method be more legitimate and conclusive than the former, I proceed now to examine; and in order thereto shall premise the following Lemma. “If with a View to demonstrate any Proposition, a certain Point is supposed, by virtue of which certain other Points are attained; and such supposed Point be it self afterwards destroyed or rejected by a contrary Supposition; in that case, all the other Points, attained thereby and consequent thereupon, must also be destroyed and rejected, so as from thence forward to be no more supposed or applied in the Demonstration.” This is so plain as to need no Proof.

XIII. Now the other Method of obtaining a Rule to find the Fluxion of any Power is as follows. Let the Quantity \( x \) flow uniformly, and be it proposed to find the Fluxion of \( x^n \). In the same time that \( x \) by flowing becomes \( x + o \), the Power \( x^n \) becomes , i. e. by the Method of infinite Series

and the Increments

are to one another as

Let now the Increments vanish, and their last Proportion will be 1 to \( n x^{n-1} \). But it should seem that this reasoning is not fair or conclusive. For when it is said, let the Increments vanish, i. e. let the Increments be nothing, or let there be no Increments, the former Supposition that the Increments were something, or that there were Increments, is destroyed, and yet a Consequence of that Supposition, i. e. an Expression got by virtue thereof, is retained. Which, by the foregoing Lemma, is a false way of reasoning. Certainly when we suppose the Increments to vanish, we must suppose their Proportions, their Expressions, and every thing else derived from the Supposition of their Existence to vanish with them.

XIV. To make this Point plainer, I shall unfold the reasoning, and propose it in a fuller light to your View. It amounts therefore to this, or may in other Words be thus expressed. I suppose that the Quantity \( x \) flows, and by flowing is increased, and its Increment I call \( o \), so that by flowing it becomes \( x + o \). And as \( x \) increaseth, it follows that every Power of \( x \) is likewise increased in a due Proportion. Therefore as \( x \) becomes \( x + o \), \( x^n \) will become that is, according to the Method of infinite Series,

And if from the two augmented Quantities we subduct the Root and the Power respectively, we shall have remaining the two Increments, to wit,

which Increments, being both divided by the common Divisor \( o \), yield the Quotients

which are therefore Exponents of the Ratio of the Increments. Hitherto I have supposed that \( x \) flows, that \( x \) hath a real Increment, that \( o \) is something. And I have proceeded all along on that Supposition, without which I should not have been able to have made so much as one single Step. From that Supposition it is that I get at the Increment of \( x^n \), that I am able to compare it with the Increment of \( x \), and that I find the Proportion between the two Increments. I now beg leave to make a new Supposition contrary to the first, i. e. I will suppose that there is no Increment of \( x \), or that \( o \) is nothing; which second
Supposition destroys my first, and is inconsistent with it, and therefore with every thing that supposeth it. I do nevertheless beg leave to retain \(nx^{n-1}\), which is an Expression obtained in virtue of my first Supposition, which necessarily presupposeth such Supposition, and which could not be obtained without it: All which seems a most inconsistent way of arguing, and such as would not be allowed of in Divinity.

XV. Nothing is plainer than that no just Conclusion can be directly drawn from two inconsistent Suppositions. You may indeed suppose any thing possible: But afterwards you may not suppose any thing that destroys what you first supposed. Or if you do, you must begin de novo. If therefore you suppose that the Augments vanish, i. e. that there are no Augments, you are to begin again, and see what follows from such Supposition. But nothing will follow to your purpose. You cannot by that means ever arrive at your Conclusion, or succeed in, what is called by the celebrated Author, the Investigation of the first or last Proportions of nascent and evanescent Quantities, by instituting the Analysis in finite ones. I repeat it again: You are at liberty to make any possible Supposition: And you may destroy one Supposition by another: But then you may not retain the Consequences, or any part of the Consequences of your first Supposition so destroyed. I admit that Signs may be made to denote either any thing or nothing: And consequently that in the original Notation \(x + o\), \(o\) might have signified either an Increment or nothing. But then which of these soever you make it signify, you must argue consistently with such its Signification, and not proceed upon a double Meaning: which to do were a manifest Sophism. Whether you argue in Symbols or in Words, the Rules of right Reason are still the same. Nor can it be supposed, you will plead a Privilege in Mathematics to be exempt from them.

XVI. If you assume at first a Quantity increased by nothing, and in the Expression \(x + o\), \(o\) stands for nothing, upon this Supposition as there is no Increment of the Root, so there will be no Increment of the Power; and consequently there will be none except the first, of all those Members of the Series constituting the Power of the Binomial: you will therefore never come at your Expression of a Fluxion legitimately by such Method. Hence you are driven into the fallacious way of proceeding to a certain Point on the Supposition of an Increment, and then at once shifting your Supposition to that of no Increment. There may seem great Skill in doing this at a certain Point or Period. Since if this second Supposition had been made before the common Division by \(o\), all had vanished at once, and you must have got nothing by your Supposition. Whereas by this Artifice of first dividing, and then changing your Supposition, you retain 1 and \(nx^{n-1}\). But, notwithstanding all this address to cover it, the fallacy is still the same. For whether it be done sooner or later, when once the second Supposition or Assumption is made, in the same instant the former Assumption and all that you got by it is destroyed, and goes out together. And this is universally true, be the Subject what it will, throughout all the Branches of humane Knowledge; in any other of which, I believe, Men would hardly admit such a reasoning as this, which in Mathematics is accepted for Demonstration.
9.1. BERKELEY’S OBJECTIONS. 1734

XVII. It may not be amiss to observe, that the Method for finding the Fluxion of a Rectangle of two flowing Quantities, as it is set forth in the Treatise of Quadratures, differs from the abovementioned taken from the second Book of the Principles, and is in effect the same with that used in the calculus differentialis. [NOTE: Analyse des Infiniment Petits, part 1. prop. 2.] For the supposing a Quantity infinitely diminished and therefore rejecting it, is in effect the rejecting an Infinitesimal; and indeed it requires a marvellous sharpness of Discernment, to be able to distinguish between evanescent Increments and infinitesimal Differences. It may perhaps be said that the Quantity being infinitely diminished becomes nothing, and so nothing is rejected. But according to the received Principles it is evident, that no Geometrical Quantity, can by any division or subdivision whatsoever be exhausted, or reduced to nothing. Considering the various Arts and Devices used by the great author of the Fluxionary Method: in how many Lights he placeth his Fluxions: and in what different ways he attempts to demonstrate the same Point: one would be inclined to think, he was himself suspicious of the justness of his own demonstrations; and that he was not enough pleased with any one notion steadily to adhere to it. Thus much at least is plain, that he owned himself satisfied concerning certain Points, which nevertheless he could not undertake to demonstrate to others. [NOTE: See Letter to Collins, Nov. 8, 1676.] Whether this satisfaction arose from tentative Methods or Inductions; which have often been admitted by Mathematicians (for instance by Dr. Wallis in his Arithmetic of Infinites) is what I shall not pretend to determine. But, whatever the Case might have been with respect to the Author, it appears that his Followers have shewn themselves more eager in applying his Method, than accurate in examining his Principles.

XVIII. It is curious to observe, what subtilty and skill this great Genius employs to struggle with an insuperable Difficulty; and through what Labyrinths he endeavours to escape the Doctrine of Infinitesimals; which as it intrudes upon him whether he will or no, so it is admitted and embraced by others without the least repugnance. Leibnitz and his followers in their calculus differentialis making no manner of scruple, first to suppose, and secondly to reject Quantities infinitely small: with what clearness in the Apprehension and justness in the reasoning, any thinking Man, who is not prejudiced in favour of those things, may easily discern. The Notion or Idea of an infinitesimal Quantity, as it is an Object simply apprehended by the Mind, hath been already considered. [NOTE: Sect. 5 and 6.] I shall now only observe as to the method of getting rid of such Quantities, that it is done without the least Ceremony. As in Fluxions the Point of first importance, and which paves the way to the rest, is to find the Fluxion of a Product of two indeterminate Quantities, so in the calculus differentialis (which Method is supposed to have been borrowed from the former with some small Alterations) the main Point is to obtain the difference of such Product. Now the Rule for this is got by rejecting the Product or Rectangle of the Differences. And in general it is supposed, that no Quantity is bigger or lesser for the Addition or Subduction of its Infinitesimal: and that consequently no error can arise from such rejection of Infinitesimals.
XIX. And yet it should seem that, whatever errors are admitted in the Premises, proportional errors ought to be apprehended in the Conclusion, be they finite or infinitesimal: and that therefore the of Geometry requires nothing should be neglected or rejected. In answer to this you will perhaps say, that the Conclusions are accurately true, and that therefore the Principles and Methods from whence they are derived must be so too. But this inverted way of demonstrating your Principles by your Conclusions, as it would be peculiar to you Gentlemen, so it is contrary to the Rules of Logic. The truth of the Conclusion will not prove either the Form or the Matter of a Syllogism to be true: inasmuch as the Illation might have been wrong or the Premises false, and the Conclusion nevertheless true, though not in virtue of such Illation or of such Premises. I say that in every other Science Men prove their Conclusions by their Principles, and not their Principles by the Conclusions. But if in yours you should allow yourselves this unnatural way of proceeding, the Consequence would be that you must take up with Induction, and bid adieu to Demonstration. And if you submit to this, your Authority will no longer lead the way in Points of Reason and Science.

XX. I have no Controversy about your Conclusions, but only about your Logic and Method. How you demonstrate? What Objects you are conversant with, and whether you conceive them clearly? What Principles you proceed upon; how sound they may be; and how you apply them? It must be remembered that I am not concerned about the truth of your Theorems, but only about the way of coming at them; whether it be legitimate or illegitimate, clear or obscure, scientific or tentative. To prevent all possibility of your mistaking me, I beg leave to repeat and insist, that I consider the Geometrical Analyst as a Logician, i.e. so far forth as he reasons and argues; and his Mathematical Conclusions, not in themselves, but in their Premises; not as true or false, useful or insignificant, but as derived from such Principles, and by such Inferences. And forasmuch as it may perhaps seem an unaccountable Paradox, that Mathematicians should deduce true Propositions from false Principles, be right in the Conclusion, and yet err in the Premises; I shall endeavour particularly to explain why this may come to pass, and shew how Error may bring forth Truth, though it cannot bring forth Science.

XXI. In order therefore to clear up this Point, we will suppose for instance that a Tangent is to be drawn to a Parabola, and examine the progress of this Affair, as it is performed by infinitesimal Differences.

Let \( AB \) be a Curve, the Abscisse \( AP = x \), the Ordinate \( PB = y \), the Difference of the Abscisse \( PM = dx \), the Difference of the Ordinate \( RN = dy \). Now by supposing the Curve to be a Polygon, and consequently \( BN \), the Increment or Difference of the Curve, to be a straight Line coincident with the Tangent, and the differential Triangle \( BRN \) to be similar to the triangle \( TPB \), the Subtangent \( PT \) is found a fourth Proportional to \( RN : RB : PB : that \) is to \( dy : dx : y \). Hence the Subtangent will be

But herein there is an error arising from the aforementioned false supposition, whence the value of \( PT \) comes out greater than the Truth: for in reality
it is not the Triangle $RNB$ but $RLB$ which is similar to $PBT$, and therefore (instead of $RN$) $RL$ should have been the first term of the Proportion, i.e. $RN + NL$, i.e. $dy + z$: whence the true expression for the Subtangent should have been

There was therefore an error of defect in making $dy$ the divisor: which error was equal to $z$, i.e. $NL$ the Line comprehended between the Curve and the Tangent. Now by the nature of the Curve $yy = px$, supposing $p$ to be the Parameter, whence by the rule of Differences $2ydy = pdx$ and

But if you multiply $y + dy$ by itself, and retain the whole Product without rejecting the Square of the Difference, it will then come out, by substituting the augmented Quantities in the Equation of the Curve, that

truly. There was therefore an error of excess in making

which followed from the erroneous Rule of Differences. And the measure of this second error is

Therefore the two errors being equal and contrary destroy each other; the first error of defect being corrected by a second error of excess.

XXII. If you had committed only one error, you would not have come at a true Solution of the Problem. But by virtue of a twofold mistake you arrive, though not at Science, yet at Truth. For Science it cannot be called, when you proceed blindfold, and arrive at the Truth not knowing how or by what means. To demonstrate that $z$ is equal to

let $BR$ or $dx$ be $m$ and $RN$ or $dy$ be $n$. By the thirty third Proposition of the first Book of the Conics of Apollonius, and from similar Triangles, as $2x$ to $y$ so is $m$ to

Likewise from the Nature of the Parabola $yy + 2yn + nn = xp + mp$, and $2yn + nn = mp$: wherefore

and because $yy = px$,

will be equal to $x$. Therefore substituting these values instead of $m$ and $x$ we shall have

i.e. which being reduced gives

XXIII. Now I observe in the first place, that the Conclusion comes out right, not because the rejected Square of $dy$ was infinitely small; but because this error was compensated by another contrary and equal error. I observe in the second place, that whatever is rejected, be it every so small, if it be real, and consequently makes a real error in the Premises, it will produce a proportional real error in the Conclusion. Your Theorems therefore cannot be accurately true, nor your Problems accurately solved, in virtue of Premises, which themselves are not accurate, it being a rule in Logic that Conclusio sequitur partem debiliorum.

Therefore I observe in the third place, that when the Conclusion is evident and the Premises obscure, or the Conclusion accurate and the Premises inaccurate, we may safely pronounce that such Conclusion is neither evident nor accurate, in virtue of those obscure inaccurate Premises or Principles; but in virtue of
some other Principles which perhaps the Demonstrator himself never knew or thought of. I observe in the last place, that in case the Differences are supposed finite Quantities ever so great, the Conclusion will nevertheless come out the same: inasmuch as the rejected Quantities are legitimately thrown out, not for their smallness, but for another reason, to wit, because of contrary errors, which destroying each other do upon the whole cause that nothing is really, though something is apparently thrown out. And this Reason holds equally, with respect to Quantities finite as well as infinitesimal, great as well as small, a Foot or a Yard long as well as the minutest Increment.

XXXI. A Point may be the limit of a Line: A Line may be the limit of a Surface: A Moment may terminate Time. But how can we conceive a Velocity by the help of such Limits? It necessarily implies both Time and Space, and cannot be conceived without them. And if the Velocities of nascent and evanescent Quantities, i. e. abstracted from Time and Space, may not be comprehended, how can we comprehend and demonstrate their Proportions? Or consider their rationes primae and ultimae? For to consider the Proportion or Ratio of Things implies that such Things have Magnitude: That such their Magnitudes may be measured, and their Relations to each other known. But, as there is no measure of Velocity except Time and Space, the Proportion of Velocities being only compounded of the direct Proportion of the Spaces, and the reciprocal Proportion of the Times; doth it not follow that to talk of investigating, obtaining, and considering the Proportions of Velocities, exclusively of Time and Space, is to talk unintelligibly?

XXXV. I know not whether it be worth while to observe, that possibly some Men may hope to operate by Symbols and Suppositions, in such sort as to avoid the use of Fluxions, Momentums, and Infinitesimals after the following manner. Suppose $x$ to be one Absciss of a Curve, and $z$ another Absciss of the same Curve. Suppose also that the respective Areas are $xxx$ and $zzz$: and that $z - x$ is the Increment of the Absciss, and $zzz - xxx$ the Increment of the Area, without considering how great, or how small those Increments may be. Divide now $zzz - xxx$ by $z - x$ and the Quotient will be $zz + zx + xx$: and, supposing that $z$ and $x$ are equal, this same Quotient will be $3xx$ which in that case is the Ordinate, which therefore may be thus obtained independently of Fluxions and Infinitesimals. But herein is a direct Fallacy: for in the first place, it is supposed that the Abscisses $z$ and $x$ are unequal, without such supposition no one step could have been made; and in the second place, it is supposed they are equal: which is a manifest Inconsistency, and amounts to the same thing that hath been before considered. [NOTE: Sect. 15.] And there is indeed reason to apprehend, that all Attempts for setting the abstruse and fine Geometry on a right Foundation, and avoiding the Doctrine of Velocities, Momentums, &c. will be found impracticable, till such time as the Object and the End of Geometry are better understood, than hitherto they seem to have been. The great Author of the Method of Fluxions felt this Difficulty, and therefore he gave in to those nice Abstractions and Geometrical Metaphysics, without which he saw nothing could be done on the received Principles; and what in the way of
Demonstration he hath done with them the Reader will judge. It must, indeed, be acknowledged, that he used Fluxions, like the Scaffold of a building, as things to be laid aside or got rid of, as soon as finite Lines were found proportional to them. But then these finite Exponents are found by the help of Fluxions. Whatever therefore is got by such Exponents and Proportions is to be ascribed to Fluxions: which must therefore be previously understood. And what are these Fluxions? The Velocities of evanescent Increments? And what are these same evanescent Increments? They are neither finite Quantities nor Quantities infinitely small, nor yet nothing. May we not call them the Ghosts of departed Quantities?

9.2 Maclaurin’s response. 1742.

Maclaurin’s treatment of fluxions. Need picture.

[45]

Maclaurin
A treatise of fluxions, 1742. II.496.

496. In general suppose, as in art. 66, that while the point P (fig. 220) describes the right line Aa with an uniform motion, the point M sets out from L with a velocity that is to the constant velocity of P as Lc to Dg, and proceeds in the right line Ee with a motion continually accelerated or retarded, that LS any space described by M is always to DG the space described in the same time by P as Lf to Dg, that cx is to Dg as the difference of the velocities of M at S and L to the constant velocity of P, and that LS is always to LC as Lf to Lc. Then LS being always expressed by LC + CS, it is manifest that (since LC is to DG as Lc to Dg, or as the velocity of M at L to the velocity of P) LC is what would have been described by M if its motion had continued uniformly from L, and that CS arises in this expression in consequence of the acceleration or retardation of the motion of the point M while it describes LS. But if LS and DG be supposed infinitely small increments of EL and AD, cx will be infinitely less than Dg; and since cf is less than cx by what was shown in art. 66, it follows that cf will be infinitely less than Lc, and CS infinitely less than LC. Therefore when the increment LS is supposed infinitely small, and its expression is resolved into two parts LC and CS, of which the former LC is always in the same ratio to DG (the simultaneous increment of AD while the increments vary, and the latter CS is infinitely less than the former LC, we may conclude that the part CS is that which arises in consequence of the variation of the motion of M while it describes LS, and is therefore to be neglected in measuring the motion of M at L, or the fluxion of the right line EL. Thus the manner of investigating the differences or fluxions of quantities in the method of infinitesimals maybe deduced from the principles of the method of fluxions demonstrated above. For instead of neglecting CS because it is infinitely less than LC (according to the usual manner of reasoning in that method), we may reject it, because we may thence conclude that it is not produced in consequence
of the generating motion at \( L \), but of the subsequent variations of this motion. And it appears why the conclusions in the method of infinitesimals are not to be represented as if they were only near the truth, but are to be held as accurately true.

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### 9.3 45b. Maclaurian's definition of limit

**Definition of Limits**

Maclaurin  
A treatise of fluxions  
1742, sections 502-503.

502. But however safe and convenient this method may be, some will always scruple to admit infinitely little quantities, and infinite orders of infinitesimals, into a science that boasts of the most evident and accurate principles as well as of the most rigid demonstrations; and therefore we chose to establish so extensive and useful a doctrine in the preceding chapters on more unexceptionable postulata. In order to avoid such suppositions, Sir Isaac Newton considers the simultaneous increments of the flowing quantities as finite, and then investigates the ratio which is the limit of the various proportions which those increments bear to each other, while he supposes them to decrease together till they vanish; which ratio is the same with the ratio of the fluxions by what was shown in art. 66, 67, and 68. In order to discover this limit, he first determines the ratio of the increments in general, and reduces it to the most simple terms, so as that (generally speaking) a part at least of each term may be independent of the value of the increments themselves; then by supposing the increments to decrease till they vanish, the limit readily appears.

503. For example, let \( a \) be an invariable quantity, \( x \) a flowing quantity, and \( o \) any increment of \( x \); then the simultaneous increments of \( xx \) and \( ax \) will be \( 2'xo + oo \) and \( ao \), which are in the same ratio to each other as \( 2x + o \) is to \( a \). This ratio of \( 2x + o \) to \( a \) continually decreases while \( o \) decreases, and is always greater than the ratio of \( 2x \) to \( a \) while \( o \) is any real increment, but it is manifest that it continually approaches to the ratio of \( 2x \) to \( a \) as its limit; whence it follows that the fluxion of \( xx \) is to the fluxion of \( ax \) as \( 2x \) is to \( a \). If \( x \) be supposed to flow uniformly, \( ax \) will likewise flow uniformly, but \( xx \) with a motion continually accelerated: the motion with which \( ax \) flows may be measured by \( ao \), but the motion with which \( xx \) flows is not to be measured by its increment \( 2xo + oo \) (by ax. 1), but by the part \( 2xo \) only, which is generated in consequence of that motion; and the part \( oo \) is to be rejected because it is generated in consequence only of the acceleration of the motion with which the variable square flows, while \( o \) the increment of its side is generated: and the ratio of \( 2xo \) to \( ao \) is that of \( 2x \) to \( a \), which was found to be the limit of the ratio of the increments \( 2xo + oo \) and \( ao \) (fig. 220).
9.4 Euler’s definition of continuity. 1748.

[ref 46]

9.5 d’Alembert’s response to Berkeley. 1754.

[ref 47]

9.6 Euler, infinitely small. 1755.

[ref 48]

82. But to return to the problem, although some may deny that an infinite number truly exists in the world, nevertheless in mathematical speculations there very often occur questions, to which, unless an infinite number is admitted, it is not possible to respond. Thus, if there is sought the sum of all the numbers that constitute this series $1 + 2 + 3 + 4 + 5 + \text{ etc.}$, since these numbers progress without end, and increase, the sum of all of them could certainly not be finite; and therefore it comes about that it must be infinite. To denote a quantity of this kind mathematicians use this sign $\infty$, by which is meant a quantity greater than any finite, or assignable, quantity. Thus when a parabola is defined in such a way that it is said to be an infinitely long ellipse, we could correctly assert that the axis of the parabola is an infinitely long straight line.

83. Moreover, this teaching on the infinite will be better illustrated, if we explain what would mathematically be infinitely small. There is no doubt that every quantity can be decreased so far that it simply vanishes, and goes to nothing. But an infinitely small quantity is nothing other than a vanishing quantity, and is therefore really equal to 0. This definition of the infinitely small also fits those quantities that are said to be less than every assignable quantity; for if a quantity becomes so small, that is less than any assignable quantity, it certainly cannot be other than nothing; for unless it were 0, it would be possible to assign a quantity equal to it, which is against the hypothesis. Therefore, to those asking what an infinitely small quantity is in mathematics, we reply that it is really 0; therefore, there are not so many mysteries hidden in this as are commonly thought, and which to many render the calculus of the infinitely small in some way suspect. Meanwhile, if they ignore them in what follows, where we teach this calculus, their doubts will be completely removed.

84. Therefore since we have shown an infinitely small quantity to be really zero, the first objection is: why do we not always denote infinitely small quantities by the sign 0, but employ special signs to denote them? For since all nothings are equal amongst themselves, it seems superfluous to denote them by different signs. But although any two zeros are equal amongst themselves, so that their difference becomes nothing, nevertheless, when the two are compared in some way, either arithmetically or geometrically, there we see a difference
between them, in fact the quantity arising from the comparison; indeed the 
arithmetical ratio between two such zeros is equality, but not the geometric ra-
tio. This is very easily seen from the geometric proportion $2 : 1 = 0 : 0$, in which 
the fourth term is 0, as is the third. Moreover from the nature of proportions, 
since the first term is twice as great as the second, it must also be that the third 
is twice as great as the fourth.

85. Moreover this is also very plain in common arithmetic; for it is known 
to everyone that zero multiplied by any number gives zero, and that $n \cdot 0 = 0$, 
and thus it will be that $n : 1 = 0 : 0$. Whence it is clear that it is possible for 
two zeros to have any geometric proportion between them, although looking at 
the thing arithmetically, their ratio is always one of equality. Therefore since it 
is possible to introduce any ratio between zeros, different symbols are used to 
indicate this diversity, since usually it is the geometric ratio between different 
zeros that is to be investigated. Indeed, in the calculus of the infinitely small 
we are concerned with nothing other than geometric ratios between different 
infinitely small quantities, and in this matter, therefore, unless we use different 
signs to indicate them, we will be led into great confusion, and with no way to 
escape from it.

86. If, therefore, as is the accepted way of writing in the analysis of infinities, 
$dx$ denotes an infinitely small quantity, we will have both $dx = 0$ and $a \, dx = 
0$, where $a$ denotes any finite quantity. This is so notwithstanding that the 
geometric ratio $a \, dx : dx$ is finite, namely as $a : 1$; and here, on account of 
that, these two infinitely small quantities $dx$ and $a \, dx$, although both are equal 
to 0, should not be confused with each other, if indeed their ratio is to be 
investigated. Similarly, if there occur different infinitely small quantities $dx$ and 
$dy$, although both are equal to 0, nevertheless their ratio is not constant. And 
to the investigation of ratios between two infinitely small quantities of this kind, 
there is applied all the power of the differential calculus. Moreover the use of 
such comparison, although it may seem at first sight very limited, is nevertheless 
found to be exceedingly broad, as from here will appear much more closely.

9.7 d'Alambert, encyclopedia

Leibniz was embarrassed by the objections he felt to exist against infinitely small 
quantities, as the appear

in the differential calculus; thus he preferred to reduce infinitely small to 
merely incomparable quantities.

...

Newton started out from another principle; and one can say that the meta-
physics of this great mathematician on the calculus of fluxions is very exact 
and illuminating, even though he allowed us only an imperfect glimpse of his 
thoughts. He never considered the differential calculus as the study of infinitely 
small quantities, but as the method of first and ultimate ratios, that is to say, 
the method of finding the limits of ratios. Thus this famous author has never
differentiated quantities but only equations; in fact, every equation involves a relation between two variables and the differentiation of equations merely consists in finding the limit of the ratio of the finite differences of the two quantities contained in the equation.

... Once this is well understood, one will feel that the assumption made concerning infinitely small quantities serves only to abbreviate and simplify the reasoning; but that the differential calculus does not necessarily suppose the existence of those quantities; and that moreover this calculus merely consists in algebraically determining the limit of a ratio, for which we already have the expression in terms of lines, and in equating those two expressions. This will provide us with one of the lines we are looking for. This is perhaps the most precise and neatest possible definition of the differential calculus; but it can be understood only when one is well acquainted with this calculus, because often the true nature of a science can understood only by those that have studies this science.

... We have seen above that in the differential calculus there are really no infinitely small quantities of the first order; that actually those quantities called $u$ are supposed to be divided by other supposedly infinitely small quantities; in this state they do not denote either infinitely small quantities or quotients of infinitely small quantities; they are the limits of the ratio of two finite quantities. This same holds for the second-order differences and for those of higher order. There is actually no quantity in Geometry such as $ddy$; whenever $ddy$ occurs in an equation it is supposed to be divided by a quantity $dx^2$, or another of the same order. What now is $ddy/dx^2$? It is the limit of the ratio $ddy/dx$ divided by $dx$; or, what is still cl

earer, it is the limit of $dz/dx$, where $dy/dx = z$ is a finite quantity.

from kant to hilbert, a source book in the foundation of mathematics, WB Ewald, pages 126-128.

9.8  d’Alembert definition of a limit. 1765.

d’Alembert

from d’Alembert and Diderot Encyclopédie, 1751–65, IX, 542.

[49]

LIMIT (Mathematics). One says that a magnitude is the limit of another magnitude, when the second may approach the first more closely than by a given quantity, as small as one wishes, moreover without the magnitude which approaches being allowed ever to surpass the magnitude that it approaches; so that the difference between such a quantity and its limit is absolutely unassignable.

For example, suppose we have two polygons, one inscribed in a circle and
the other circumscribed; it is clear that one may increase the number of sides as much as one wishes, and in that case each polygon will approach ever more closely to the circumference of the circle; the perimeter of the inscribed polygon will increase and that of the circumscribed polygon will decrease, but the perimeter or edge of the first will never surpass the length of the circumference, and that of the second will never be smaller than that same circumference; the circumference of the circle is therefore the limit of the increase of the first polygon and of the decrease of the second.

1. If two magnitudes are the limit of the same quantity, the two magnitudes will be equal to each other.

2. Suppose $A \times B$ is the product of two magnitudes $A$, $B$. Let us suppose that $C$ is the limit of the magnitude $A$, and $D$ the limit of the quantity $B$; I say that $C \times D$, the product of the limits, will necessarily be the limit of $A \times B$, the product of the magnitude $A, B$.

These two propositions, which one will find demonstrated exactly in the Institutions de Géométrie, serve as principles for demonstrating rigorously that one has the area of a circle from multiplying its semicircumference by its radius. See the work cite, p. 331 and following in the second volume.

The theory of limits is the foundation of the true justification of the differential calculus. See differential, fluxion, exhaustion, infinite. Strictly speaking, the limit never coincides, or never becomes equal to the quantity of which it is the limit, but the latter approaches it ever more closely, and may differ from it as little as one wishes. The circle, for example, is the limit of the inscribed and circumscribed polygons; for strictly it never coincides with them, although they may approach it indefinitely. This notion may serve to clarify several mathematical propositions. For example, one says that the sum of a decreasing geometric progression in which the first term is $a$ and the second is $b$, is $a^n - b$; this value is never strictly the sum of the progression, it is the limit of that sum, that is to say, the quantity which it may approach as closely as one wishes, without ever arriving at it exactly. For if $e$ is the last term in the progression, the exact value of the sum is $a^n - b$, which is always less than $a^n - b$ because even in a decreasing geometric progression, the last term $e$ is never 0; but as this term continually approaches zero, without ever arriving at it, it is clear that zero is its limit, and that consequently the limit of $\frac{a^n - b}{a - b}$ is $\frac{a^n}{a - b}$, supposing $e = 0$, that is to say, on putting in place of $e$ its limit.

[ref 49]

9.9 Euler’s introduction to integration. 1768

[ref 50]
9.10 Lagrange ‘avoids’ infinitesimals, on arbitrariness small intervals. 1797.

[ref 51, 51b]

9.11 Bolzano, greatest lower bound, continuity. 1817.

[ref 52, 53]

9.12 Cauchy, limites, intermediate value theorem, continuity. 1821.

[54abc]

9.13 Cauchy - limits, derivative, integral, FTC. 1823.

[55abcd]


Stendall, pages 450–452. [56b]

GBH Riemann, 1854.

On the concept of a definite integral and the extent of its validity

The uncertainty that still prevails on some fundamental points of the theory of definite integrals requires us to begin with something about the concept of a definite integral and the extent of its validity.

So, first: What is one to understand by \( \int_a^b f(x) \, dx \)?

To establish this, we take a sequence of values \( x_1, x_2, \ldots, x_{n-1} \), following one after another between \( a \) and \( b \) in order of size, and for the sake of brevity we denote \( x_1 - a \) by \( \delta_1 \), \( x_2 - x_1 \) by \( \delta_2 \), \( \ldots \), \( b - x_{n-1} \) by \( \delta_n \), and by \( \epsilon \) a proper fraction. Then the value of the sum

\[
S = \delta_1 f(a + \epsilon_1 \delta_1) + \delta_2 f(x_1 + \epsilon_2 \delta_2) + \delta_3 f(x_2 + \epsilon_3 \delta_3) + \cdots + \delta_n f(x_{n-1} + \epsilon_n \delta_n)
\]
will depend on the choice of the intervals $\delta$ and the quantities $\epsilon$. If this now has the property that, however $\delta$ and $\epsilon$ are chosen, it comes infinitely close to a fixed limit $A$ when all the $\delta$ become infinitely small, then this value is called by $\int_{a}^{b} f(x) \, dx$.

If it does not have this property, then $\int_{a}^{b} f(x) \, dx$ has no meaning. But even then, there have been several attempts to attribute a meaning to this symbol, and among these extensions of the concept of a definite integral there is one accepted by all mathematicians. Namely, if the function $f(x)$ become infinitely large when the variable approaches a particular value $c$ in the interval $(a, b)$, then clearly the sum $S$, no matter what order of smallness one ascribes to $\delta$, can take any arbitrary value; thus it has no limiting value, and $\int_{a}^{b} f(x) \, dx$ according to the above would have no meaning. But if then

$$\int_{a}^{c-\alpha_{1}} f(x) \, dx + \int_{c+\alpha_{2}}^{b} f(x) \, dx$$

approaches a fixed limit, as $\alpha_{1}$ and $\alpha_{2}$ become infinitely small, then one understands by this limit $\int_{a}^{b} f(x) \, dx$. [ref 56ab]


[ref 57]

[ref 52, 53]

9.16 Cauchy, limites, intermediate value theorem, continuity. 1821.

[54abc]

9.17 Cauchy - limits, derivative, integral, FTC. 1823.

[55abcd]

9.18 Riemann. The integral. 1854.

Stendall, pages 450–452. [56b]

GBH Riemann, 1854.

On the concept of a definite integral and the extent of its validity
The uncertainty that still prevails on some fundamental points of the theory of definite integrals requires us to begin with something about the concept of a definite integral and the extent of its validity.

So, first: What is one to understand by \( \int_a^b f(x) \, dx \)?

To establish this, we take a sequence of values \( x_1, x_2, \ldots, x_{n-1} \), following one after another between \( a \) and \( b \) in order of size, and for the sake of brevity we denote \( x_1 - a \) by \( \delta_1 \), \( x_2 - x_1 \) by \( \delta_2 \), \ldots, \( b - x_{n-1} \) by \( \delta_n \), and by \( \epsilon \) a proper fraction. Then the value of the sum

\[
S = \delta_1 f(a + \epsilon_1 \delta_1) + \delta_2 f(x_1 + \epsilon_2 \delta_2) + \delta_3 f(x_2 + \epsilon_3 \delta_3) + \cdots + \delta_n f(x_{n-1} + \epsilon_n \delta_n)
\]

will depend on the choice of the intervals \( \delta \) and the quantities \( \epsilon \). If this now has the property that, however \( \delta \) and \( \epsilon \) are chosen, it comes infinitely close to a fixed limit \( A \) when all the \( \delta \) become infinitely small, then this value is called by \( \int_a^b f(x) \, dx \).

If it does not have this property, then \( \int_a^b f(x) \, dx \) has no meaning. But even then, there have been several attempts to attribute a meaning to this symbol, and among these extensions of the concept of a definite integral there is one accepted by all mathematicians. Namely, if the function \( f(x) \) become infinitely large when the variable approaches a particular value \( c \) in the interval \((a, b)\), then clearly the sum \( S \), no matter what order of smallness one ascribes to \( \delta \), can take any arbitrary value; thus it has no limiting value, and \( \int_a^b f(x) \, dx \) according to the above would have no meaning. But if then

\[
\int_a^{c - \alpha_1} f(x) \, dx + \int_{c + \alpha_2}^b f(x) \, dx
\]

approaches a fixed limit, as \( \alpha_1 \) and \( \alpha_2 \) become infinitely small, then one understands by this limit \( \int_a^b f(x) \, dx \). [ref 56ab]

[ref 57]
Chapter 10

Issues to resolve

• Too many readings. Don’t want too straight a route. But can’t be too long.

• How to bring out the variety of problems calculus addresses?

• Make sure to connect to issues of continuity and real numbers, Dedekind.

• How to make tangents seem more ‘natural’

• tangents change from ‘that which touches’ to ‘secant limits’. How/when? Motion along curves?

• should we do more of Descartes, so to have more curves to work with? Cycloid. Folium of Descartes?
Chapter 11

Sources


[Th] Greek Mathematical Works, Igor Thomas

[Str] A Source Book in Mathematics 1200-1800, Struik, our library

[Sm] Source book in Mathematics, Smith, our library

[Ca] Classics of Mathematics, Calinger, our library

[Ste] Mathematics Emerging, A Sourcebook, our library

[SJC] numbered are from the SJC manual

[BrB] Bnumbered are from Br. Brendan

[E] Edwards The Historical Development of the Calculus


[Cl] Nicole Oresme and the medieval geometry of qualities and motions, Translated and Edited by Marshall Clagett, Madison, University of Wisconsin Press, 1968


Chapter 12

Not Used

12.1 Squaring the Circle

12.1.1 Aristophanes, *The Birds*, 414 B.C.

METON: I have come to you.
PISTHETAERUS: Yet another pest! What have you come to do? What’s your plan? What’s the purpose of your journey? Why these splendid buskins?
METON: I want to survey the plains of the air for you and to parcel them into lots.
PISTHETAERUS: In the name of the gods, who are you?
METON: Who am I? Meton, known throughout Greece and at Colonus.
PISTHETAERUS: What are these things?
METON: Tools for measuring the air. In truth, the spaces in the air have precisely the form of a furnace. With this bent ruler I draw a line from top to bottom; from one of its points I describe a circle with the compass. Do you understand?
PISTHETAERUS: Not the very least.
METON: With the straight ruler I set to work to inscribe a square within this circle; in its centre will be the market-place, into which all the straight streets will lead, converging to this centre like a star, which, although only orbicular, sends forth its rays in a straight line from all sides.
PISTHETAERUS: Meton, you new Thales. ²

12.1.2 Aristotle, *Physics Book I Part 2*, 185a, 14-17. date??:

Moreover, no man of science is bound to solve every kind of difficulty that may be raised, but only as many as are drawn falsely from the principles of the science: it is not our business to refute those that do not arise in this way: just as it is the

² From http://www.gutenberg.org/files/3013/3013.txt
duty of the geometer to refute the squaring of the circle by means of segments, but it is not his duty to refute Antiphon's proof.

12.1.3 Plutarch, *On Exile* c. 100:

There is no place that can take away the happiness of a man, nor yet his virtue or wisdom. Anaxagoras, indeed, wrote on squaring of the circle while in prison.

12.1.4 Themistius: Commentary on Aristotle’s *Physics* c. 340?

For such false arguments as preserve the geometrical hypothesis are to be refuted by geometry, but such as conflict with them are to be left alone. Examples are given by two men who tried to square the circle, Hippocrates of Chios and Antiphon. The attempt of Hippocrates is to be refuted. For, while preserving the principles, he commits a paralogism by squaring only that lune which is described about the side of the square inscribed in the circle, though including every lune that can be squared in the proof. But the geometer could have nothing to say against Antiphon, who inscribed an equilateral triangle in the circle, and on each of the sides set up another triangle, an isosceles triangle with its vertex on the circumference of the circle, and continued this process, thinking that at some time he would make the side of the last triangle, although a straight line, coincide with the circumference.

12.1.5 Proclus, *A Commentary on the First Book of Euclid’s Elements*, c. 450?:

I think it was in consequence of this problem [Euclid I.45] that the ancient geometers were led to investigate the squaring of the circle. For if a parallelogram is found equal to any rectilineal figure, it is worth inquiring whether it be not also possible to prove rectilineal figures equal to circular. Archimedes in fact proved that any circle is equal to a right-angled triangle wherein one of the sides about the right-angle is equal to the radius and the base to the perimeter.

12.1.6 Simplicius, Commentary on Aristotle’s *Physics*, c. 530

Antiphon described a circle and inscribed some one of the (regular) polygons that can be inscribed therein. Suppose, for example, that the inscribed polygon is a square. ... It is clear that the breach with the principles of geometry comes about not, as Alexander says, “because the geometer lays down as a hypothesis that a circle touches a straight line in one point [only], while Antiphon violates this.” For the geometer does not lay this down as a hypothesis, but it is proved

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3 *On Exile* 17, 607E,F [Th, vol II, page 309], c. 100
4 [Th Vol 2]. Also SJC
5 [Th vol 2, 317]
in the third book of the Elements. It would be better therefore to say that the principle is that a straight line cannot coincide with the circumference, a straight line drawn from outside the circle touching it in one point only, a straight line drawn from inside cutting it in two points and not more, and tangential contact being in one point only. Now continual division of the space between the straight line and the circumference of the circle will never exhaust it nor ever reach the circumference of the circle, if the space is really divisible without limit. For if the circumference could be reached, the geometrical principle that magnitudes are divisible without limit would be violated. This was the principle which Eudemus says was violated by Antiphon.

12.2 Tangents to Spirals

Archimedes On spirals. Constructed tangents to them. Reference? Is this readable?

What to do about this???

[Th Vol 2]. Also, SJC.