Functioning in the Complex Plane

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1 Abstract

In this paper we will explore preliminary topics in complex analysis. We will define what complex numbers, complex valued functions, limits of complex functions, and derivatives of complex functions, and will prove some of the interesting properties of each. We will conclude with a brief survey of applications of complex analysis.

2 The Complex numbers

2.1 The Construction of the Complex Numbers

The study of complex numbers began in the 16th century with Italian mathematicians investigating the solutions to cubic polynomials found solutions involving the square roots of negative numbers. The mathematician Giorlarno Cardano studied the simultaneous equations $x + y = 10$ and $xy = 40$. He encountered the solution

$$x = 5 + \sqrt{-15} \text{ and } y = 5 - \sqrt{-15}$$

Girolamo Cardano
(1501-1576)

He saw that if the usual algebraic rules were assumed then the equations were in fact satisfied but he disregarded this result as being "as refined as it is useless".Additionally Cardano commented on equations of the form

$$x^3 = 3px + 2q$$

Whose solutions represents the intersection of a cubic polynomial and the line $3px + 2q$. The Tartaglia formula states that

$$x = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}}$$

gives a solution to such equations. But if $q^2 > p^3$ then the solution involves square roots of negative numbers. Cardano noted in particular that $x^3 = 15x + 4$ gave the solution

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

It wasn't until the 18th century that complex numbers gained wide use, and it took 300 years from their conception to create a definition equivalent to the modern definition.

Definition 1. *A complex number is a number with real and imaginary components which we write as $z = a + ib$, where $a, b \in \mathbb{R}$ and $i^2 = -1$. a is the real component of the complex number and ib is called the imaginary component because $i^2 = -1$ has no solutions in the real numbers. If $a = 0$, then we call z a purely imaginary number, and if $b = 0$ then we call z trivially real.*

From this definition we can see that any real number is a trivially complex number. For the complex number z we denote the real component as $Re(z) = a$ and the imaginary component as $Im(z) = b$. The term complex number was not originally meant to indicate extra intricacy but was instead meant to reflect its composite nature, being a combination of a real and an imaginary number.

Definition 2. *The set of complex numbers is defined to be*

$$\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}\}$$

It wasn't until operations on complex numbers were defined that complex numbers to be taken seriously. Thirty years after Cardano published his thoughts on roots of cubic polynomials the mathematician Bombelli continued his work. Bombelli found that if he assumed imaginary numbers worked algebraically like real numbers then

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} = 4$$

which is an actual solution to $x^3 = 15x + 4$

This observation lead to the definitions for the following three operations for any two complex numbers $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$.

1. For addition of complex numbers we add each of the components (real and imaginary) of the complex numbers separately. For the complex numbers $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ this can be written as $\mathrm{Re}(z_1 + z_2) = \mathrm{Re}(z_1) + \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1 + z_2) = \mathrm{Im}(z_1) + \mathrm{Im}(z_2)$, or $z_1 + z_2 = (a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2)i$.

2. Multiplication of complex number is carried out by assuming the distributive and multiplying out as you would for real numbers: $(a_1 + b_1 i)(a_2 + b_2 i) = a_1 a_2 + a_1 b_2 i + a_2 b_1 i + b_1 b_2 i^2 = (a_1 a_2 - b_1 b_2) + (b_1 a_2 + a_1 b_2)i$.

Figure 1: Bombelli was the first to define operations on complex numbers.

We can see that many of the basic properties of real numbers are still true for the complex numbers.

Theorem 1. *(Properties of Complex Operations)*
For any complex numbers z_1, z_2, z_3

1. *Addition is commutative;* $z_1 + z_2 = z_2 + z_1$.
2. *Multiplication is commutative;* $z_1 z_2 = z_1 z_2$.
3. *Addition is associative;* $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$.
4. *Multiplication is associative;* $z_1(z_2 z_3) = (z_1 z_2) z_3$.

Proof. We obtain the proofs for each of these properties through direct calculation. Let $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$, $z_3 = a_3 + b_3 i$ be arbitrary complex numbers.
1.

$$\begin{aligned} z_1 + z_2 &= (a_1 + b_1 i) + (a_2 + b_2 i) \\ &= (a_1 + a_2) + (b_1 + b_2) i \\ &= (a_2 + a_1) + (b_2 + b_1) i \\ &= (a_2 + b_2 i) + (a_1 + b_1 i) \\ &= z_2 + z_1 \end{aligned}$$

2.

$$\begin{aligned} z_1 z_2 &= (a_1 + b_1 i)(a_2 + b_2 i) \\ &= (a_1 a_2 - b_1 b_2) + (b_1 a_2 + a_1 b_2) i \\ &= (a_2 a_1 - b_2 b_1) + (b_2 a_1 + a_2 b_1) i = z_2 z_1 \end{aligned}$$

3.

$$
\begin{aligned}
& z_1 + (z_2 + z_3) \\
&= (a_1 + b_1 i) + ((a_2 + b_2 i) + (a_3 + b_3 i)) \\
&= (a_1 + b_1 i) + ((a_2 + a_3) + (b_2 + b_3) i) \\
&= (a_1 + (a_2 + a_3)) + (b_1 + (b_2 + b_3)) i \\
&= ((a_1 + a_2) + a_3) + ((b_1 + b_2) + b_3) i \\
&= ((a_1 + a_2) + (b_1 + b_2) i) + (a_3 + b_3 i) \\
&= ((a_1 + b_1 i) + (a_2 + b_2 i)) + (a_3 + b_3 i) \\
&= (z_1 + z_2) + z_3
\end{aligned}
$$

4.

$$
\begin{aligned}
& (z_1 z_2) z_3 \\
&= ((a_1 + b_1 i)(a_2 + b_2 i))(a_3 b_3 i) \\
&= ((a_1 a_2 - b_1 b_2) + (b_1 a_2 + a_1 b_2) i)(a_3 + b_3 i) \\
&= ((a_1 a_2 - b_1 b_2)(a_3) - (b_1 a_2 + a_1 b_2)(b_3)) + ((b_1 a_2 + a_1 b_2) a_3 + (a_1 a_2 - b_1 b_2) b_3) i \\
&= (a_1 a_2 a_3 - b_1 b_2 a_3) - (b_1 a_2 b_3 + a_1 b_2 b_2) + (b_1 a_2 a_3 + a_1 b_2 a_3 + a_1 a_2 b_3 - b_1 b_2 b_3) i \\
&= (a_1 (a_2 a_3 - b_2 b_3) - b_1 (b_2 a_3 + a_2 b_3)) + (b_1 (a_2 a_3 - b_2 b_3) + a_1 (b_2 a_3 + a_2 b_3)) i \\
&= (a_1 + b_1 i)((a_2 a_3 - b_2 b_3) + (b_2 a_3 + a_2 b_3) i) \\
&= (a_1 + b_1 i)((a_2 + b_2 i)(a_3 + b_3 i)) \\
&= z_1 (z_2 z_3)
\end{aligned}
$$

□

2.2 The Complex Plane

To understand complex numbers intuitively we need alternative ways to represent them. We can think of the complex numbers in a geometric sense if we associate each complex number $a + bi$ with the point (a, b) in the xy-plane.We then refer to this plane as the *complex plane*. We call the x-axis the *real axis* and the y axis the *imaginary axis*. We can see an example of four complex numbers depicted in the complex plane in figure 2. This geometric representation allows us to relate complex numbers to trigonometry and geometry.

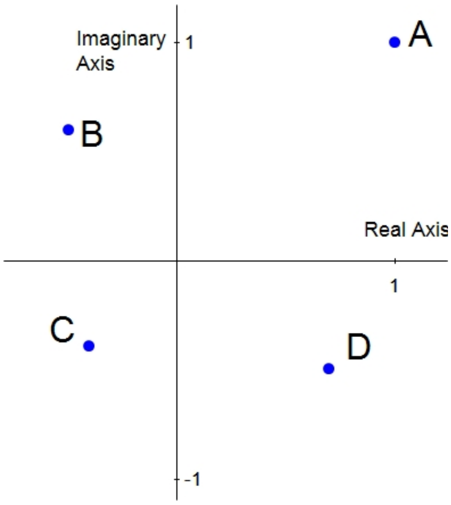

Figure 2: Four complex numbers depicted in the complex plane. $A = 1+i$, $B = -.5+.7i$, $C = -.4 - .4i$, $D = .7 - .5i$

Equivalently we can relate the complex numbers to vectors by associating each complex number $z = a + bi$ with the vector created by drawing a directed line segment from the origin to the point (a, b) in the complex plane, as shown in figure 3.

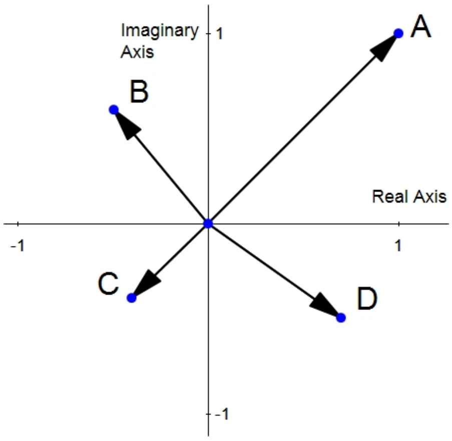

Figure 3: The complex numbers from figure 3 depicted as vectors in the complex plane

Any complex number can also be put into a trigonometric form. For the complex number $z = a + bi$ we can form a right triangle in the complex plane from the origin and the points $(a, b), (a, 0)$, as depicted in Figure 4. From this we can see that any complex number can be expressed in the form:

$$z = r(\cos\theta + i\sin\theta)$$

We call this the *polar form* of z. We call θ the *argument* of z. The argument of a complex number is not unique, and if a is an argument of the complex number Z then $a + t\pi$ is also an argument of z for all $t \in \mathbb{Z}$. The value for which $-\pi < \theta \leq \pi$ is called the *principal argument* of z, denoted arg$z$.

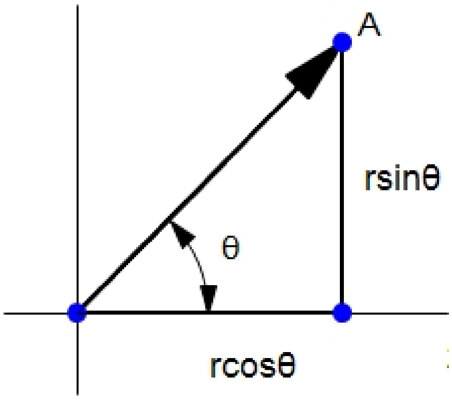

Figure 4: The construction of the polar form of a complex number

The *absolute value* or *modulus* of a complex number is the real number $|z| = \sqrt{a^2 + b^2}$. $|z|$ is the distance in the complex plane between the complex number $|z|$ and the origin. The value $|z_1 - z_2|$ is equal to the distance between the two points z_1 and z_2. Illustration of absolute value to go here.

The *complex conjugate* of a complex number z is $\overline{z} = a - bi$. The complex conjugate of a complex number is its reflection across the real axis. A trivially complex number is its own conjugate. In figure 5 we can see two complex conjugates. $A = 2 + i$ is the complex conjugate of $B = 2 - i$ and B is also the complex conjugate of A.

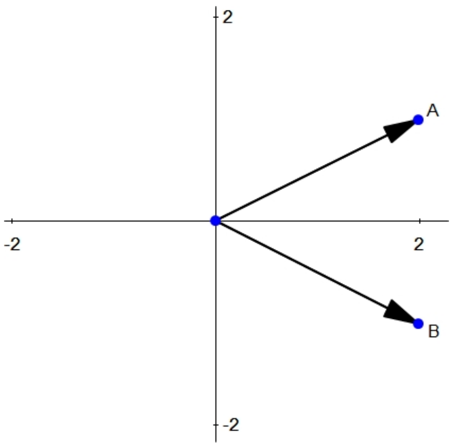

Figure 5: Two complex conjugates

We can derive some very useful properties which we will soon use from these two definitions.

Theorem 2. *(Properties of Complex Conjugate and Modulus)*
For any complex numbers z_1, z_2, z_3

1. $|z_1 z_2| = |z_1||z_2|$
2. $(a)|z| = |\overline{z}|$, $(b) z\overline{z} = |z|^2$
3. $\overline{z_1 \pm z_2} = \overline{z}_1 \pm \overline{z}_2$
4. $\overline{\overline{z}} = z$
5. $(a) Re(z) = \frac{z+\overline{z}}{2}$, (b) $Im(z) = \frac{z-\overline{z}}{2i}$
6. $Re(z_1 z_2) \leq |z_1 z_2|$

Proof. Let $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ be arbitrary complex numbers.
1.

$$\begin{aligned}|z_1 z_2| &= |(a_1 + b_1 i)(a_2 + b_2 i)| = |(a_1 a_2 - b_1 b_2) + (b_1 a_2 + a_1 b_2)i| \\ &= \sqrt{(a_1 a_2 - b_1 b_2)^2 + (b_1 a_2 + a_1 b_2)^2} \\ &= \sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)} = \sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2} = |z_1||z_2|\end{aligned}$$

2. For the complex number $z = a + bi$ we can see that

$$\begin{aligned}&(a)|\overline{z}| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z| \\ &(b) z\overline{z} = (a + bi)(a - bi) = (a^2 + b^2) + (ba - ab) = a^2 + b^2 = |z|^2\end{aligned}$$

3. For the complex numbers $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ it is apparent that

$$\begin{aligned}\overline{z_1 + z_2} &= \overline{(a_1 + a_2) + (b_1 + b_2)i} \\ &= (a_1 + a_2) - (b_1 + b_2)i \\ &= (a_1 - b_1 i) + (a_2 - b_2 i) \\ &= \overline{z_1} + \overline{z_2}\end{aligned}$$

4. For the complex number $z = a + bi$ we can see that

$$\begin{aligned}\overline{\overline{z}} &= \overline{\overline{a + bi}} \\ &= \overline{a - bi} = a - (-bi) = a + bi = z\end{aligned}$$

5.

$$\begin{aligned}(a)\frac{z + \overline{z}}{2} &= \frac{(a + bi) + (a - bi)}{2} = \frac{2a}{2} = a = \text{Re}(z) \\ (b)\frac{z - \overline{z}}{2} &= \frac{(a + bi) - (a - bi)}{2} = \frac{2bi}{2} = bi = \text{Im}(z)\end{aligned}$$

6.

$$\begin{aligned}\text{Re}(z_1z_2) &= (a_1a_2 - b_1b_2)\\ &=\sqrt{(a_1a_2 - b_1b_2)^2}\\ &\leq\sqrt{(a_1a_2 - b_1b_2)^2 + (b_1a_2 + a_1b_2)^2}\\ &= |z_1z_2|\end{aligned}$$

□

The complex conjugate also allows us to derive a division operation for complex numbers. We define the quotient of two complex numbers $z_1 = a_1 + b_1i$ and $z_2 = a_2 + b_2i$ by

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{a_1 + b_1i}{a_2 + b_2i}\\ &= \frac{a_1 + b_1i}{a_2 + b_2i}\frac{a_2 - b_2i}{a_2 - b_2i}\\ &= \frac{(a_1a_2 + b_1b_2) + i(b_1a_1 - a_1b_2)}{a_2^2 + b_2^2}\\ &= \frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} + \frac{b_1a_1 - a_1b_2}{a_2^2 + b_2^2}i\end{aligned}$$

Theorem 3. *The triangle inequality, an important property of many metrics, holds for the modulus of a complex number. That is,*

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

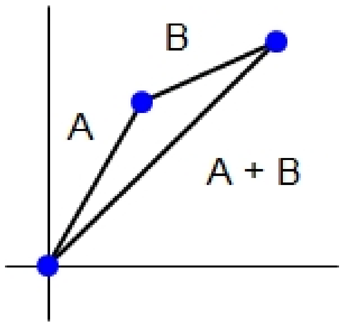

Figure 6: The Triangle inequality

The proof of this theorem follows geometrically from the fact that no side of a triangle has greater length than the sum of the lengths of the other two sides, as shown in figure 6, but we will show this algebraically using theorem 2.

Proof. Let $z_1, z_2 \in \mathbb{C}$ be arbitrary. We begin by applying Theorem #2(b) to $z = z_1 + z_2$

$$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2}) = (z_1 + z_2)(\overline{z}_1 + \overline{z}_2)$$

By the definition of multiplication of complex numbers

$$(z_1 + z_2)(\overline{z}_1 + \overline{z}_2) = z_1\overline{z}_1 + z_1\overline{z}_2 + z_2\overline{z}_1 + z_2\overline{z}_2$$

using Thm#2b, $z\overline{z} = |z|^2$ again

$$z_1\overline{z}_1 + z_1\overline{z}_2 + z_2\overline{z}_1 + z_2\overline{z}_2 = |z_1|^2 + z_1\overline{z}_2 + z_2\overline{z}_1 + |z_2|^2$$

By Thm#4, along with the commutativity of multiplication of complex numbers.

$$|z_1|^2 + z_1\overline{z}_2 + z_2\overline{z}_1 + |z_2|^2 = |z_1|^2 + (z_1\overline{z}_2 + \overline{\overline{z_1}z_2}) + |z_2|^2$$

By Thm#5(a)

$$|z_1|^2 + (z_1\overline{z}_2 + \overline{\overline{z_1}z_2}) + |z_2|^2 = |z_1|^2 + 2\text{Re}(z_1\overline{z}_2) + |z_2|^2$$

And finally, by Thm#6, followed by factoring

$$|z_1|^2 + 2\text{Re}(z_1\overline{z}_2) + |z_2|^2 \leq |z_1|^2 + 2|z_1z_2| + |z_2|^2 = (|z_1| + |z_2|)^2.$$

We have shown that

$$(|z_1 + z_2|)^2 \leq (|z_1| + |z_2|)^2$$

which implies

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

□

3 Functions in the Complex Plane

In this section we will investigate properties of complex valued functions. We will prove several theorems whose real valued analogues were central to Real Analysis.

3.1 Topology of the Complex plane

To begin with we must define terminology related to the topology of the complex plane, which will be central to the definition of a limit of a complex function.

Definition 3. *Let $\epsilon > 0$ be an arbitrary real number. The ϵ-neighborhood of a point $z_0 \in \mathbb{C}$ is the set of all complex numbers which satisfy the inequality $|z - z_0| < \epsilon$. That is $\{z \mid |z - z_0| < \epsilon\}$*

In figure 7 we see an example ϵ-neighborhood of the point $5 - 2i$ with $\epsilon = .4$.

We call a point $z_0 \in S$ an *interior point* of the set S if and only if there exists some ϵ-neighborhood of z_0 which is a subset of S. Conversely we call z_0 an *exterior point* of the set S if and only if there is some ϵ-neighborhood of z_0 which contains no points in S.

A *limit point* or *accumulation point* of a set of complex numbers S is a point for which every ϵ-neighborhood of z_0 contains an infinite number of points in S.

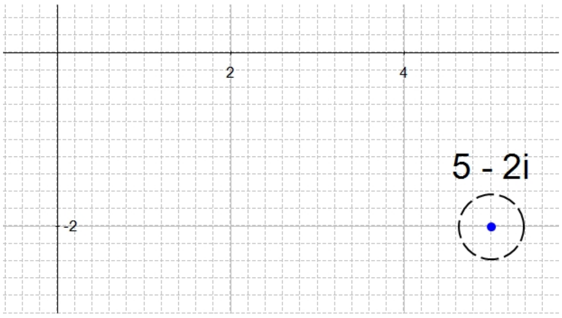

Figure 7: The ϵ-neighborhood of $5 - i$, for $\epsilon = .4$.

3.2 Complex Functions

We define a complex function similarly to how we define a real valued function.

Definition 4 (Complex Functions)**.** *A complex function f defined on a set of complex numbers S is a rule which assigns to each $z = x + iy$ in S a unique complex number $w = u + iv$ which is written as $f : S \to \mathbb{C}$*

From this definition we can see that a complex function maps each complex number in a set of complex numbers to one to one and only one other complex number. The set of complex numbers which the function does map, S in the definition, is called the *domain* of f and the complex numbers which are mapped to $\{f(z) : z \in S\}$ is called the *range* or *image* of S. In figure 8 we see an example. If the domain of $f(z) = z^2$ is the single point $1 + i$ then the range is the single point $2i$.

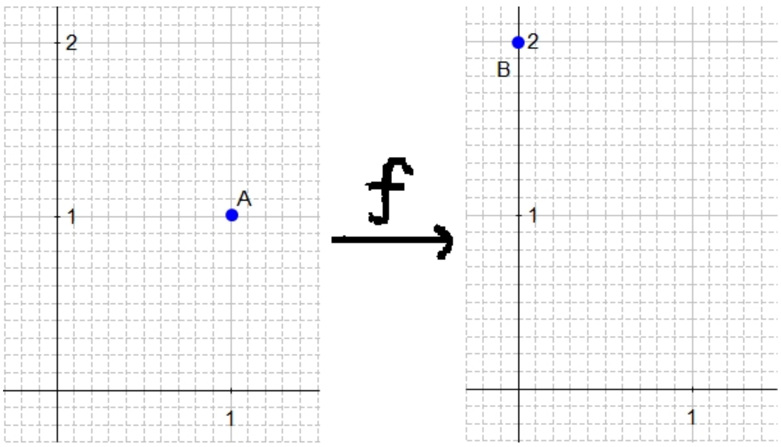

Figure 8: The complex function $f(z) = z^2$ maps the point A = (1,1), depicted on the left to the point B = (2,0), shown on the right.

We now have the necessary foundation to define the limit of a complex function.

Definition 5 (Limit of a Complex Function)**.** *For a function f defined at all points in some ϵ-neighborhood of z_0, with the possible exception of the point z_0, we call the complex number L the limit of f(z) as z approaches z_0 and write $L = \lim_{z \to z_0} f(z)$ if and only if for any $\epsilon > 0$, we can find $\delta > 0$ such that*

$$0 < |z - z_0| < \delta \Rightarrow |f(z) - L| < \epsilon$$

This definition says that L is the limit of $f(z)$ as z approaches z_0 if whenever you are given an $\epsilon > 0$ you can find an δ-neighborhood around z_0 such that every point within the δ-neighborhood maps to a point the ϵ-neighborhood around z_0. Some examples will make this definition clearer.

Limit Example 1 $\lim\limits_{z \to 2-i} (2z + 1) = 5 - 2i$
Let $\epsilon > 0$. Let $\delta = \frac{\epsilon}{2}$ and suppose that $|z - (2 - i)| < \delta$. Then

$$\begin{aligned} |f(z) - L| &= |(2z + 1) - (5 - 2i)| \\ &= |(2z + 2i - 4)| \\ &= |2(z + i - 2)| \\ &= 2|z + i - 2| \\ &= 2|z - (2 - i)| \\ &< 2\frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

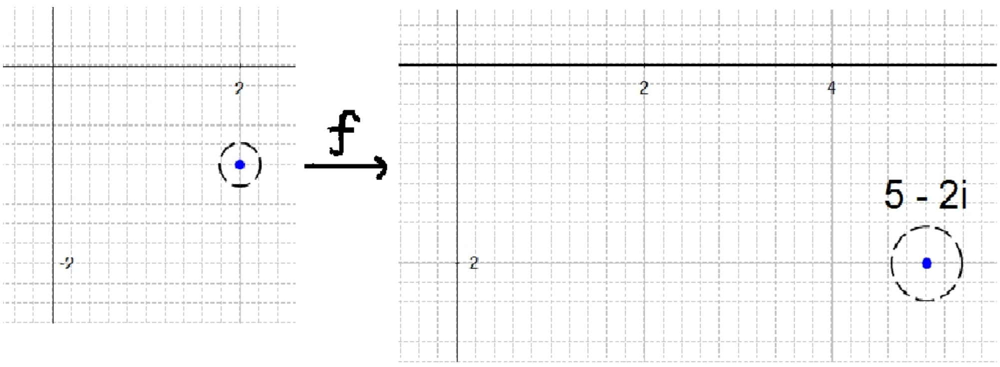

Figure 9: An illustration of example 1. For the function $f(z) = 2z + 1$ all points in the δ-neighborhood depicted on the left map to points within the ϵ-neighborhood on the right.

Limit Example 2: $\lim\limits_{z \to z_0} (z^2 + 5) = {z_0}^2 + 5$
Let $\epsilon > 0$, $z_0 \in \mathbb{C}$, and $\delta = \min\{\frac{\epsilon}{3|z_0|}, |z_0|\}$. Suppose that $|z - z_0| < \delta$.. If we add $2|z_0|$ to

$|z - z_0| < |z_0|$ we can see that $|z + z_0| < 3|z_0|$. Observe that:

$$\begin{aligned}
|f(z) - L| &= |f(z) - ({z_0}^2 + 5| \\
&= |(z^2 + 5) - ({z_0}^2 + 5)| \\
&= |z^2 - {z_0}^2| \\
&= |z + z_0||z - z_0| \\
&< 3|z_0||z - z_0| \\
&< 3|z_0|\frac{\epsilon}{3|z_0|} \\
&= \epsilon
\end{aligned}$$

From the definition of a complex limit it it natural to ask if a function can have more than one limit at any point. We find that the intuitive answer, no, is correct.

Theorem 4. ***The Uniqueness of Limits of Complex Valued Functions***
The limit of a complex valued function is unique. That is, for any function $f(z)$, if $\lim_{z \to z_o} f(z) = L_1$ and $\lim_{z \to z_o} f(z) = L_2$ then $L_1 = L_2$.

Proof. Let $f : S \to \mathbb{C}$ where $S \subseteq \mathbb{C}$. Suppose that $f(z)$ has two distinct limits, L_1 and L_2, as z goes to z_0. Let $\epsilon = |\frac{l_1 - l_2}{2}$—. Then we know that there exists δ_1, δ_2 such that if $|z - z_0| < \delta_1$ then $|f(z) - l_1| < \epsilon$ and if $|z - z_0| < \delta_2$ then $|f(z) - l_2| < \epsilon$.
Let $\delta = \min\{\delta_1, \delta_2\}$. Then if $|z - z_0| < \delta$

$$|f(z) - L_1| < \epsilon \text{ and } |f(z) - L_2| < \epsilon$$

Then we can see by the triangle inequality that:

$$|L_1 - L_2| = |L_1 - f(z) + f(z) - L_2| \leq |L_1 - f(z)| + |f(z) - L_2|$$

And by our definition of z

$$|L_1 - f(z)| + |f(z) - L_2| < \epsilon + \epsilon = \left|\frac{L_1 - L_2}{2}\right| + \left|\frac{L_1 - L_2}{2}\right|$$

This is a contradiction and therefore the limit of a complex valued function is unique. □

Next we will look at some commonly used properties of complex numbers which have strikingly similar analogues for properties of limits of real numbers. When we are interested in the limit of a complicated function, these properties may allow us to break the function down into smaller, more manageable parts.

Theorem 5. *If $\lim\limits_{z \to z_0} f(z) = L_1$ and $\lim\limits_{z \to z_0} g(z) = B$, then*

1. $\lim_{z \to z_0}(f(z) + g(z)) = L_1 \pm L_2$
2. $\lim_{z \to z_0}(f(z)g(z)) = L_1 L_2$

Proof. Let $f(z)$ and $g(z)$ be complex functions such that $\lim_{z\to z_0} f(z) = L_1$ and $\lim_{z\to z_0} g(z) = L_2$.
Part 1: Let $\epsilon > 0$. Then there exists $\delta_1 > 0$ such that

$$\text{if } |z - z_0| < \delta_1 \text{ then } |f(z) - L_1| < \frac{\epsilon}{2}$$

Similarly there exists $\delta_2 > 0$ such that

$$\text{if } |z - z_0| < \delta_2 \text{ then } |g(z) - L_2| < \frac{\epsilon}{2}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then if $|z - z_0| < \delta$:

$$|(f(z) + g(z)) - (L_1 + L_2)| \le |f(z) - L_1| + |g(z) - L_2| < \epsilon/2 + \epsilon/2 = \epsilon$$

Part 2: Let $\epsilon > 0$. Then there exists $\delta_1 > 0$ such that

$$\text{if } |z - z_0| < \delta_1 \text{ then } |f(z) - L_1| < \left|\frac{\epsilon}{2(1 + L_2)}\right|.$$

since l_1 is the limit of $f(z)$ as z approaches z_0.
Similarly there exists $\delta_2 > 0$ such that

$$\text{if } |z - z_0| < \delta_2 \text{ then } |g(z) - L_2| < \left|\frac{\epsilon}{2(1 + L_1}\right|.$$

by the same reasoning.
We also know that there exists some $\delta_3 > 0$ such that

$$\text{if } |z - z_0| < \delta_3 \text{ then } |f(z) - L_1| < 1$$

which implies $|f(z)| < 1 + L_1|$.
Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then if $|z - z_0| < \delta$

$$\begin{aligned}
|f(z_0)g(z_0) - L_1L_2| &= |f(z_0)g(z_0) + f(z_0)L_2 - f(z_0)L_2 - L_1L_2| \\
&\le |f(z_0)g(z_0) - f(z_0)L_2| + |f(z_0)L_2 - L_1L_2| \\
&= |f(z_0)||g(z_0) - L_2| + |L_2||f(z_0) - L_1| \\
&< |f(z_0)|\left|\frac{\epsilon}{2(1 + L_1}\right| + |L_2|\left|\frac{\epsilon}{2(1 + L_2)}\right| \\
&< |1 + L_1|\left|\frac{\epsilon}{2(1 + L_1}\right| + \left|1 + L_2||\frac{\epsilon}{2(1 + L_2)}\right| \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

□

For many functions the limit at a point is equal to the functional value at that point. We call such functions *continuous* at that point. More technically, for a function $f(z)$ defined in an ϵ-neighborhood of the point z_0 the function is continuous at the point z_0 if and only if $\lim_{z\to z_0} f(z) = f(z_0)$. We call f continuous on a set S if it is continuous at every point in S.

Theorem 6. *If $f(z)$ and $g(z)$ are both continuous at $z_0 \neq 0$ then $f(z) \pm g(z)$ and $f(z)g(z)$ are both continuous at z_0.*

Proof. Suppose $f(z)$ and $g(z)$ are both continuous at $z_0 \neq 0$. Then $\lim\limits_{z\to z_0} f(z) = f(z_0)$ and $\lim\limits_{z\to z_0} g(z) = g(z_0)$
1. By Theorem 3 we know that $\lim_{z\to z_0}(f(z)+g(z)) = f(z_0)+g(z_0)$, which implies $f(z)+g(z)$ is continuous at z_0. 2. Similarly by theorem 3 we know that $\lim_{z\to z_0} f(z)g(z) = f(z_0)g(z_0)$, implies that $f(z)g(z)$ is continuous at z_0. □

4 Complex Derivatives

After defining the limit of a complex function we are now able to dfine its derivative. A core question we might have about a complex function is "what is its rate of change?" To find a complex functions rate of change between two points we divide the amount of change by the distance between the two points. The average rate of change of a function, $f(z)$, between two points, z_1 and z_2 is equal to:

$$\frac{|f(z_2) - f(z_1)|}{|z_2 - z_1|}$$

A harder question to answer is "what is the instantaneous rate of change?", that is the rate of change at a single point. To estimate the instantaneous rate of change at a point z_1 then we evaluate the rate of change between z_1 and $z_1 + \delta z$ for succesively smaller δz. Then our rate of change equation becomes

$$\frac{|f(z_1) - f(z_1 + \delta z)|}{|\delta z|}$$

As δz decreases we will generally get better estimates,although it is not guaranteed. To obtain the exact answer we take the limit as δz approaches 0. This leads to the following definition.

Definition 6 (Derivative)**.** *For a function f defined in a neighborhood of a point z_0, we define the derivative of f at z_0 by*

$$\frac{df}{dz}(z_0) = f'(z_0) = \lim_{\delta z\to 0} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z}$$

Theorem 7. *If f and g are both differentiable at a point z_0 then*

1. $(f \pm g)'(z_0) = f'(z_0) + g'(z_0)$

2. $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$

Proof. 1. We can see that

$$\begin{aligned}
(f \pm g)'(z_0) &= \lim_{\delta z \to 0} \frac{(f \pm g)(z_0 + \delta z) - (f \pm g)(z_0)}{\delta z)} \\
&= \lim_{\delta z \to 0} \frac{f(z_0 + \delta z) \pm g(z_0 + \delta z) - (f(z_0) \pm g(z_0)}{\delta z)} \\
&= \lim_{\delta z \to 0} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z)} \pm \lim_{\delta z \to 0} \frac{g(z_0 + \delta z) - g(z_0)}{\delta z)} \\
&= f'(z_0) \pm g'(z_0)
\end{aligned}$$

2. By using theorem 3 repeatedly we can see that

$$\begin{aligned}
(fg)'(z_0) &= \lim_{\delta z \to 0} \frac{(fg)(z_0 + \delta z) - (fg)(z_0)}{\delta z} \\
&= \lim_{\delta z \to 0} \frac{f(z_0 + \delta z)g(z_0 + \delta z) - f(z_0)g(z_0)}{\delta z} \\
&= \lim_{\delta z \to 0} \frac{f(z_0 + \delta z)g(z_0 + \delta z) + (f(z)g(z) - f(z)g(z)) - f(z_0)g(z_0)}{\delta z} \\
&= \lim_{\delta z \to 0} \frac{f(z_0)[g(z_0 + \delta z) - g(z_0)] + g(z_0 + \delta z)[f(z_0 + \delta z) - f(z_0)]}{\delta z} \\
&= \lim_{\delta z \to 0} f(z_0)\frac{g(z_0 + \delta z) - g(z_0)}{\delta z_0} + \lim_{\delta z \to 0} g(z_0 + \delta z)\frac{f(z_0 + \delta z) - f(z_0)}{\delta z} \\
&= [\lim_{\delta z \to 0} f(z_0)][\lim_{\delta z \to 0} \frac{g(z_0 + \delta z) - g(z_0)}{\delta z_0}] \\
&+ [\lim_{\delta z \to 0} g(z_0 + \delta z)][\lim_{\delta z \to 0} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z}] \\
&= f(z_0)g'(z_0) + g(z_0)f'(z_0)
\end{aligned}$$

□

Because the limit of a complex function is defined using an ϵ-neighborhood the direction in which z approaches z_0 does not matter. We may take advantage of this to derive some necessary conditions for a complex function to be differentiable at a point.

Theorem 8. *(The Cauchy-Riemann Relations) If a complex function $f(z) = u(x, y) + iv(x, y)$ is differentiable at the point $z_0 = x_0 + iy_0$ then:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

at the point z_0.

Proof. Let $f(z) = u(x, y) + iv(x, y)$ be a complex function which is differentiable at the point $z_0 = x_0 + iy_0$. Then when computing the limit

$$f'(z_0) = \lim_{\delta z \to 0} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z}$$

δz may approach zero in any manner and we will obtain the same results since the definition of a limit is independent of path. If δz approaches zero horizontally, that is along the real axis, then $\delta z = \delta x$, and therefore

$$\begin{aligned}
f'(z_0) &= \lim_{\delta z \to 0} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z)} \\
&= \lim_{\delta x \to 0} \frac{u(x_0 + \delta x, y_o) + iv(x_0 + \delta x, y_0) - u(x_0, y_o) - iv(x_0, y_0)}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{u(x_0 + \delta x, y_o) - u(x_0, y_o)}{\delta x} + i \lim_{\delta x \to 0} \frac{v(x_0 + \delta x, y_0) - v(x_0, y_0}{\delta x} \\
&= \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0)
\end{aligned}$$

where $\frac{\partial f}{\partial a}$ is the partial derivative of f with respect to a.
If δz approaches zero vertically, that is along the imaginary axis, then $\delta z = i\delta y$, and therefore

$$\begin{aligned}
f'(z_0) &= \lim_{\delta z \to 0} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z} \\
&= \lim_{\delta y \to 0} \frac{u(x_0, y_o + \delta y) + iv(x_0, y_0 + \delta y) - u(x_0, y_o) - iv(x_0, y_0)}{i\delta y} \\
&= \lim_{\delta y \to 0} \frac{u(x_0, y_o + \delta y) - u(x_0, y_o)}{i\delta y} + i \lim_{\delta y \to 0} \frac{v(x_0, y_0 + \delta y) - v(x_0, y_0)}{i\delta y} \\
&= \frac{1}{i} \lim_{\delta y \to 0} \frac{u(x_0, y_o + \delta y) - u(x_0, y_o)}{\delta y} + i \frac{1}{i} \lim_{\delta y \to 0} \frac{v(x_0, y_0 + \delta y) - v(x_0, y_0)}{\delta y} \\
&= \frac{1}{i} \frac{\partial u}{\partial y}(x_0, y_0) + i \frac{1}{i} \frac{\partial v}{\partial y}(x_0, y_0) \\
&= -i \frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0)
\end{aligned}$$

If we equate the real an imaginary parts of each equation we obtain the desired results:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

□

We have completed our tour of preliminary complex analysis. A natural continuation of this work would be to define complex integration, introduce power series representations of complex functions, and derive Euler's equation. We now continue on to a brief survey of applications of complex analysis.

5 Application of Complex Analysis

Complex analysis is widely used in physics, engineering and other mathematical fields. Several of these applications are outlined below.

Physics and Engineering

Complex numbers are useful for the modeling of waves because waves have both phase and amplitude. To model both of these quantities in one equation one can introduce complex numbers. Because waves are essential to many physics fields one can apply complex analysis to the examples below. Complex numbers are not necessary for studying any of these applications, but instead act as a tool which simplifies problems.

In signal processing, a commonly encountered type of signal is the complex waveform. A complex waveform is a summation of waves with different frequencies and amplitudes. By using a *Fourier transform* one can obtain the amplitude of a particular frequency in a complex waveform. If this is done for all possible frequencies one obtains the equation for a given complex waveform.

The Fourier transform has widespread uses, one of the most commonly used applications is the mp3 file format. Audio is a complex waveform and by analyzing the frequencies and amplitudes of an audio signal you can recreate the audio signal. Essentially the Fourier transform encodes how loud each particular note is in a sound. This file format uses much less storage space than previous methods of modeling audio, hence its ubiquity.

Some of the other physics and engineering fields which complex numbers are powerful tools for are electrical engineering, thermodynamics, hydrodynamics, quantum mechanics, and optics.

Kramers-Kroning Relations

The Kramers-Kroning relations connect the real and imaginary components of any complex function which is differentiable at all points where the imaginary component is greater than or equal to zero. This has far-reaching physical implications. For a large category of physics functions called response functions, these relations allow you to understand the state of a system by measuring the energy which has dissipated from it. The most widespread example of this is in electron spectroscopy.

The Riemann zeta function

The Riemann zeta function is the complex valued function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

which has been extensively studied and is the foundation for a branch of number theory called analytic number theory. It leads to the prime number theorem which describes the distribution of prime numbers among the integers.

Fundamental Theorem of Algebra

The Fundamental Theorem of Algebra states that any single-variable polynomial, with degree n and complex coefficents, has exactly n complex roots. Because all real numbers are trivially complex this also applies to all single-variable polynomials with real coefficients.

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