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The Banach-Tarski Paradox

Author:

Alex Lowen

Supervisor:

Professor Conner

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The good Christian should beware of mathematicians. The danger already exists that mathematicians have made a covenant with the devil to darken the spirit and confine man in the bonds of Hell.

-- *St. Augustine*

Introduction

According to Wapner's "The Pea and the Sun " [Wap05], there are three fundamental philosophies concerning what is true in mathematics. The first is constructivism; that is, nothing is true until one can construct it. For example, constructivists tend to believe that irrational numbers do not exist because we cannot find an algorithm that can compute one for us. Furthermore, a perfect circle is a nice idea, but one cannot construct a 100% perfect circle and so such a thing does not exist according to the constructivists. A more middle ground philosophy, according to Wapner, is formalism. Formalists deal with the structure of logic built on fundamental assumptions called axioms. Instead of proving these assumptions are true, Formalists measure an axiom's validity based on contradictions that develop from said axiom. Platonism, is the very optimistic third philosophy. Specifically, Platonists believe mathematical truth exists much like any object we can see. The fact that we can perceive a concept in some manner is sufficient proof that some part of said concept is true.

Platonism has inspired several new ideas originally viewed with skepticism. We will be addressing one such inspired idea, called Modern Set Theory. It is used everywhere in mathematics today; however, that did not use to be the case. Modern Set Theory was riddled with contradictions and paradoxes and the Banach-Tarski paradox is one such paradox.

Historical Context

Modern Set Theory was invented by Georg Cantor in the early 20^{th} century. At the time, most mathematicians were constructivists or formalists, due to the introduction of finite-automata and computers. Modern Set Theory only made sense with a Platonist mind-set, which was regarded as an out of date philosophy concerning mathematics.

The purpose of Modern Set Theory is to work with the infinite by containing an infinite number of objects in a singularity called a set. It was Georg Cantor that came up with the proof that some infinities are bigger than others, along with how one can determine

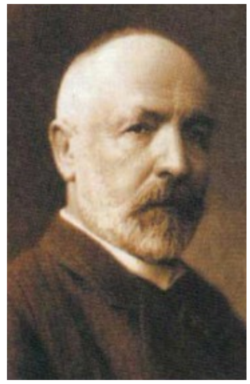


Figure 1: Geog Cantor: The inventor of Modern Set Theory

the size of an infinite set. However, Modern Set Theory faced many contradictions, such as Russell's paradox, that ruined his reputation while alive. The ridicule was so strong that Cantor eventually gave up on mathematics altogether. Working with the infinite was mocked by most mathematicians at the turn of the 20^{th} century and the paradoxes gave greater incentive for a constructivist view point. According to Wapner, Georg Cantor worked as a professor at an unremarkable university far below his potential. Leopold Kronecher avidly opposed Georg Cantor's ideas and Georg blamed Kronecher for much of his hardship, for losing the opportunity to work at the University of Berlin.

Georg Cantor spent most of his life trying to prove the Continuum Hypothesis, one of the great 20^{th} century problems, to validate Modern Set Theory. He was unsuccessful with his attempts and the rising criticism of his ideas eventually caused him to give up on mathematics all together.

Figure 2: Leopold Kronecker: The adversary to Cantor's Progress

Yet, his ideas about Modern Set Theory have revolutionized mathematics in the 20^{th} century though it was not until after he died that Modern Set Theory became widely used. After Georg Cantor's death, Zermelo and Fraenkel came up with axioms for Modern Set Theory that mimicked the axioms Euclid gave to geometry. These axioms resolved a lot of the paradoxes Georg Cantor was plagued with.

Despite the success of these axioms, one concept, used implicitly for decades before becoming an axiom, turned out to be very problematic. It is called the Axiom of Choice. In 1914, Banach and Tarski proved that with the Axiom of Choice, one could duplicate objects in space just by cutting them up, moving or rotating the pieces around, and gluing them back together. Their work was based on an incomplete rendition done by Vitali and Hausdorff.

The Banach-Tarski paradox caused much panic amongst mathematicians. Several, including Russell, believed that there was something fundamentally wrong with logic itself. After several years of panic and consideration, most mathematicians have come to accept the Banach-Tarski paradox as inevitable and adapted to it accordingly. There is even a subject that developed called Measure Theory to help reconcile the Banach-Tarski paradox.

We will be addressing how one proves the Banach-Tarski paradox as well as how it depends on the Axiom of Choice.

Figure 3: Ernst Zermelo: The founder of the Axioms of Modern Set Theory

Groups, Free Groups, Group Actions, and Partitions

We start our discussion by considering the three dimensional euclidean space called $\mathbb{R}^3$. The Banach-Tarksi paradox informally states the following:

The Banach-Tarski Paradox. *The unit sphere can be cut up into pieces, rotated around, and reassembled in a new way to create two new unit spheres.*

However, we will be addressing the formal Banach-Tarski Paradox using the language of mathematics.

The Banach-Tarski Paradox. *S^2 is SO_3-paradoxical.*

In order to prove the Banach-Tarkski Paradox, we will need to go over some preliminary concepts regarding free groups, group actions, and partitions.

An important algebraic structure we will be working with is called a group. A group is a set equipped with a binary operation along with a few additional properties. We address what a binary operation is along with what these additional properties are in modest detail.

Definition 1. Let X be a set. We call a function $*$ defined by $*\colon X \times X \to X$ a **binary operation** on X.

Definition 2. Let G be a set equipped with a binary operation $*$ on G. If for every g_1, g_2, and g_3 in G the following properties hold:

1. $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$, (Associativity)
2. There is an $e \in G$ such that $e * g_1 = g_1 * e = g_1$, (Identity)
3. There is a $g_1^{-1} \in G$ such that $g_1 * g_1^{-1} = g_1^{-1} * g_1 = e$, (Inverses)

then G is a **group**.

There are four remarks to be made about this definition. First, due to definition [**?**], we have for every g and g' in G that $g * g' \in G$. This is called the closure property. Second, due to the identity property, a group cannot be empty. Third, these properties of groups are sufficient to gaurantee e is unique and g_1^{-1} is unique for each g_1. Finally, a non-empty subset of G that has the closure and inverses properties is called a *subgroup* of G.

The reason we are interested in groups though is due to group actions. These are functions that represent the notion of elements of G transforming elements into other elements of the set, without the elements of G necessarily being functions themselves. We formalize this notion next.

Definition 3. Let G be a group with identity e and with binary operation $*$. Furthermore, let X be a set. For every $x \in X$ and for every g and h in G, A (left) **group action** is a function $F : G \times X \to X$ that satisfies the following two properties:

1. $F(e, x) = x$ (Identity)

2. $F(g * h, x) = F(g, F(h, x))$. (Compatibility)

Here are a couple of examples of group actions.

Example 1. Invertible 3×3 matrices with real entries equipped with normal matrix multiplication turns out to be a group[1] and is denoted by $GL_3(\mathbb{R})$. This group acts on $\mathbb{R}^3$ by the matrix-vector product $F\colon GL_3(\mathbb{R}) \times \mathbb{R}^3 \to \mathbb{R}^3$. That is, for every $A \in GL_3(\mathbb{R})$ and $v \in \mathbb{R}^3$ denoted as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ and } v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The corresponding group action mapping is:

$$F(A, v) = \begin{bmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \\ a_{31}x + a_{32}y + a_{33}z \end{bmatrix}.$$

[1]See Michael Artin's Algebra [Art91].

Example 2. If G is a group equipped with the operation $*$, then G acts on itself via left multiplication. That is, for every g and g' in G, the corresponding group action mapping is $F(g, g') = g * g'$.

Sometimes we find that when group G acts on a set X, there is a point $x \in X$ such that for some non-identity $g \in G$, $F(g, x) = x$. We call this point a *non-trivial fixed point* of G.

Notation. Instead of writing, $F(g, x)$ we often use $g(x)$ instead. Furthermore, unless otherwise stated, if a and b are elements of a group, then ab will be used to denote $a * b$.

In the context of the Banach Tarski paradox, the group actions we care about come from a type of group called a free group. In order to precisely define what a free group is, we need a few preliminary definitions.

Definition 4. Let G be a group. A **word** w in G is an expression (possibly, empty) in the elements of G. Specifically,

$$w = g_1^{\epsilon_1} \dots g_n^{\epsilon_n}, \text{ where } g_i \in G \text{ and } \epsilon_i \in \{1, -1\}$$

We write the empty word as the symbol ϵ.

Definition 5. A word w in a group G defined by

$$w = g_1^{\epsilon_1} \dots g_n^{\epsilon_n}, \text{ where } g_i \in G \text{ and } \epsilon_i \in \{1, -1\}$$

is **reduced** if $g_i^{\epsilon_i} \neq g_{i+1}^{-(\epsilon_{i+1})}$ for all $i = 1, \dots, n-1$. That is, a reduced word cannot contain the sub-words xx^{-1} or $x^{-1}x$ for any $x \in G$.

Definition 6. Let G be a group and suppose $X \subseteq G$. The **subgroup generated by X** is the set $\langle X \rangle$ denoted by

$$\{x_1^{\epsilon_1} \dots x_n^{\epsilon_n} : x_i \in X, \epsilon_i \in \{1, -1\}, n \in \mathbb{N}\}.$$

That is, $\langle X \rangle$ is the set of all words in G composed only with elements found in $X \cup \{x^{-1} \colon x \in X\}$.

Definition 7. A group G is called a **free group** if there exists a set X of G such that $\langle X \rangle = G$ and every non-empty reduced word in $\langle X \rangle$ defines a non-identity element of G. The cardinality of X is called the **rank** of G.

We will show that there is a free group that acts on the unit sphere. But first, we need to clarify where this free group comes from.

Definition 8. The group of all rotations about the origin of three-dimensional Euclidean space $\mathbb{R}^3$ under the operation of composition is called the **special orthogonal group** in three dimensions, or $\mathbf{SO_3}$ for short.

Notation. Since all rotations are linear transformations, it follows that every rotation in SO_3 can be represented by a 3×3 orthogonal matrix[2]. Implicitly, every matrix representation will be in respect to the standard basis elements $(1,0,0)^T, (0,1,0)^T$ and $(0,0,1)^T$, where T is the transpose of a matrix or vector.

Example 3. *Orthogonal matrices preserve vector length, so the action of Example 1 restricted to SO_3 also restricts to S^2.*

Theorem 1. *There is a subgroup of SO_3 that is free of rank 2.*

Proof. Let SO_3 act on the unit sphere S^2 and consider the rotations θ and ϕ of a point on S^2 with representations

$$\phi = \begin{bmatrix} \frac{1}{3} & -\frac{2\sqrt{2}}{3} & 0 \\ \frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2\sqrt{2}}{3} \\ 0 & -\frac{2\sqrt{2}}{3} & \frac{1}{3} \end{bmatrix}$$

[2]This fact comes from linear algebra and is far too removed from the scope of this exposition to justify here. For a complete justification, see Michael Artin's book on Algebra [Art91].

Likewise, we can create representations of the rotations θ^{-1} and ϕ^{-1} and in general,

$$\phi^{\pm 1} = \begin{bmatrix} \frac{1}{3} & \mp\frac{2\sqrt{2}}{3} & 0 \\ \pm\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \theta^{\pm 1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & \pm\frac{2\sqrt{2}}{3} \\ 0 & \mp\frac{2\sqrt{2}}{3} & \frac{1}{3} \end{bmatrix}$$

We claim $\langle \phi, \theta \rangle$ is a free group of rank 2. To prove this, we will show no non-empty reduced words are the identity. Consider the point $(1,0,0)^T \in S^2$. We will show that $w(1,0,0)^T \neq (1,0,0)^T$ for all reduced words that do not fix (1,0,0). We do this by proving that $w(1,0,0)^T = (a, b\sqrt{2}, c)^T/3^k$ for integers a, b, and c. Then, it suffices to show that $3 \nmid b\sqrt{2}$ and so $b\sqrt{2} \neq 0$ and hence $w(1,0,0)^T \neq (1,0,0)^T$. It is easily checked that $\theta^{\pm 1}(1,0,0)^T = (1,0,0)^T$. Hence, the only non-fixed actions on the point $(1,0,0)^T$ using only one element is $\phi^{\pm 1}$. We find that

$$\phi^{\pm 1}(1,0,0)^T = \begin{bmatrix} \frac{1}{3} & \mp\frac{2\sqrt{2}}{3} & 0 \\ \pm\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ \pm 2\sqrt{2} \\ 0 \end{bmatrix}$$

which fits the form $(a, b\sqrt{2}, c)^T/3^k$ where $a = 1$, $b = \pm 2\sqrt{2}$, $c = 0$ and $k = 1$. Next, by the induction hypothesis, suppose that $w'(1,0,0)^T$ is of the form $(a', b'\sqrt{2}, c')/3^{k-1}$. There are two cases. If we apply $\phi^{\pm 1}$ to $w'(1,0,0)^T$, we get

$$\phi^{\pm 1}(a', b'\sqrt{2}, c')^T/3^{k-1} = \frac{1}{3^{k-1}} \begin{bmatrix} \frac{1}{3} & \mp\frac{2\sqrt{2}}{3} & 0 \\ \pm\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a' \\ b'\sqrt{2} \\ c' \end{bmatrix} = \frac{1}{3^k} \begin{bmatrix} a' \mp 4b' \\ (b' \pm 2a')\sqrt{2} \\ 3c' \end{bmatrix}$$

which means a, b, and c defined by $a = a' \mp 4b'$, $b = b' \pm 2a'$, and $c = 3c'$ are all integers and thus $\phi^{\pm}w'(1,0,0)$ is of the form $(a, b\sqrt{2}, c)/3^k$.

If instead, we apply $\theta^{\pm 1}$ to $w'(1,0,0)^T$, we get

$$\theta^{\pm 1}(a', b'\sqrt{2}, c')^T/3^{k-1} = \frac{1}{3^{k-1}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & \pm\frac{2\sqrt{2}}{3} \\ 0 & \mp\frac{2\sqrt{2}}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} a' \\ b'\sqrt{2} \\ c' \end{bmatrix} = \frac{1}{3^k} \begin{bmatrix} 3a' \\ (b' \pm 2c')\sqrt{2} \\ c' \mp 4b' \end{bmatrix}$$

which means a, b, and c defined by $a = 3a'$, $b = b' \pm 2c'$, and $c = c' \mp 4b'$ are all integers and thus $\theta^{\pm} w'(1, 0, 0)$ is of the form $(a, b\sqrt{2}, c)/3^k$. This completes the induction argument.

Next we show by cases that the integer b in $(a, b\sqrt{2}, c)/3^k$ can never be divisible by 3 for all irreducible words that aren't trivial. We note that $\phi^{\pm 1}\theta^{\pm 1}(\text{v})$, $\theta^{\pm 1}\phi^{\pm 1}(v)$, $\phi^{\pm 1}\phi^{\pm 1}(v)$, and $\theta^{\pm 1}\theta^{\pm 1}(v)$ accounts for all possible words, where v is some arbitrary word, possibly the identity.

Case 1. Consider the first four cases $\phi^{\pm 1}\theta^{\pm 1}(v)$, depending on the signs. Let

$$v = (a'', b''\sqrt{2}, c'')/3^{k-2},$$
$$\theta^{\pm 1}(v) = (3a'', (b'' \pm 2c'')\sqrt{2}, c'' \mp 4b'')3^{k-1} = (a', b'\sqrt{2}, c')/3^{k-1},$$
$$\phi^{\pm 1}\theta^{\pm 1}(v) = (a' \pm 4b', (b' \pm 2a')\sqrt{2}, 3c')/3^k = (a, b\sqrt{2}, c)/3^k.$$

We see that $b = b' \pm 2a'$. But we know that $3 \mid a'$ because $3a'' = a'$ and so $2a'$ is a multiple of 3. This means that if $3 \nmid b'$ then $3 \nmid b$ $b \equiv b' \pmod 3$.

Case 2. Similarly, consider the next four cases $\theta^{\pm 1}\phi^{\pm 1}(v)$, let

$$v = (a'', b''\sqrt{2}, c'')/3^{k-2},$$
$$\phi^{\pm 1}(v) = (a'' \mp 4b'', (b'' \pm 2a'')\sqrt{2}, 3c'')3^{k-1} = (a', b'\sqrt{2}, c')/3^{k-1},$$
$$\theta^{\pm 1}\phi^{\pm 1}(v) = (3a', (b' \pm 2c')\sqrt{2}, c' \mp 4b')/3^k = (a, b\sqrt{2}, c)/3^k.$$

Instead, we see that $b = b' \pm 2c'$. But this time we know that $3 \mid c'$ because $3c'' = c'$ and so $2c'$ is a multiple of 3. That is, if $3 \nmid b'$ then $3 \nmid b$ since $b \equiv b' \pmod 3$.

Case 3. Now we deal with the slightly more complicated four cases of $\phi^{\pm 1}\phi^{\pm 1}(v)$. Let

$$v = (a'', b''\sqrt{2}, c'')/3^{k-2},$$
$$\phi^{\pm 1}(v) = (a'' \pm 4b'', (b'' \pm 2a'')\sqrt{2}, 3c'')3^{k-1} = (a', b'\sqrt{2}, c')/3^{k-1},$$
$$\phi^{\pm 1}\phi^{\pm 1}(v) = (a' \pm 4b', (b' \pm 2a')\sqrt{2}, 3c')/3^{k} = (a, b\sqrt{2}, c)/3^{k}.$$

From this, we gather that $b = b' \pm 2a' = b' \pm 2(a'' \mp 4b'') = b' + b'' \pm 2a'' - 9b'' = 2b' - 9b''$. Since $3 \mid -9b''$, it suffices to show that $3 \nmid 2b'$. So long as $3 \nmid b'$, a 3 will not be found in its unique prime factorization. We see that $2b'$ only adds a 2 to b's unique prime factorization. Hence, $3 \nmid 2b'$. Since $b \equiv 2b' \pmod 3$, we conclude that if $3 \nmid b'$, then $3 \nmid b$.

Case 4. Lastly, consider the similar four cases $\theta^{\pm 1}\theta^{\pm 1}v$. Let

$$v = (a'', b''\sqrt{2}, c'')/3^{k-2},$$
$$\theta^{\pm 1}(v) = (3a'', (b'' \pm 2a'')\sqrt{2}, c'' \pm 4b'')3^{k-1} = (a', b'\sqrt{2}, c')/3^{k-1},$$
$$\theta^{\pm 1}\theta^{\pm 1}(v) = (3a', (b' \pm 2a')\sqrt{2}, c' \pm 4b')/3^{k} = (a, b\sqrt{2}, c)/3^{k}.$$

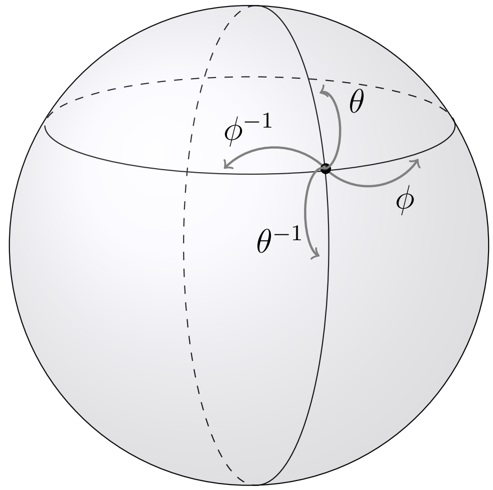
Likewise, we gather that $b = b' \pm 2c' = b' \pm (c'' \mp 4b'') = b' + b'' \pm 2c'' - 9b'' = 2b' - 9b''$. By the same reasoning as before, if $3 \nmid b'$ then $3 \nmid b$.

Since all sixteen cases have been considered, we conclude by induction that $b \neq 0$ if and only if w does not fix $(1, 0, 0)^T$. This same argument can be repeated with $(0, 0, 1)^T$ to complete the proof.

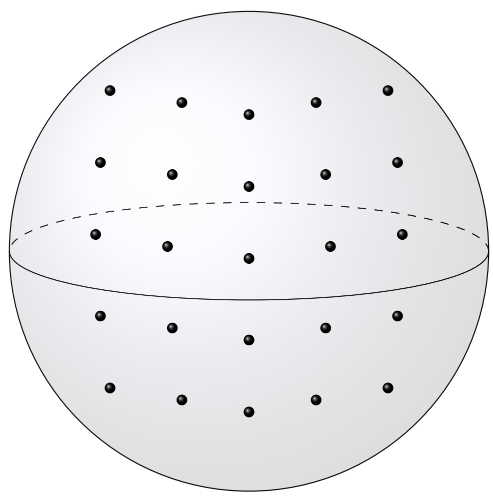
Now that we have shown the above F is a free subgroup of rank 2, we will use F to cut the unit sphere into pieces. The following definitions and theorems will be useful to us when showing what pieces of the unit sphere are needed to duplicate the sphere.

Definition 9. Let G be a group that acts on a set X. Given $x \in X$, the **G-orbit** of x is the set $\mathcal{O}_x$ defined by $\mathcal{O}_x = \{g(x) \colon g \in G\}$. Furthermore, the collection of all G-orbits of X is defined as $\{\mathcal{O}_x \colon x \in X\}$.

Example 4. Suppose we have a point on the unit sphere. Furthermore, let G act on this point via vertical and horizontal rotations as such:



The orbit of this point is all the points accessible to it via actions that can be written as combinations of $\theta, \phi, \theta^{-1}$, and ϕ^{-1}. Here is a helpful, but potentially misleading illustration.



Here are some additional things to consider about orbits that this illustration cannot showcase. First, orbits can have an infinite number of points, in this example; however, I am unable to draw an infinite number of points. Second, for every two points on the sphere, there could be a point in between. Thus, the points in an orbit can be *dense* on S^2. So all the empty space that can be seen between points may not actually be empty space.

Even though there are an infinite number of points in an orbit on the unit sphere, most of the points still are inaccessible via these rotations. This is because the number of points on the sphere is *uncountable*, whereas the number of points in the orbit is *countable*[3]. Despite this, the union of every distinct orbit makes up the whole sphere.

Definition 10. Let X be a set and let P be the collection $\{X_\alpha \subseteq X\}_{\alpha \in \Lambda}$, where Λ is some indexing set. We say that the elements of P collectively **partition** X if $X_\alpha \neq \emptyset$, $X = \bigcup_{\alpha \in \Lambda} X_\alpha$, and if $X_\alpha \cap X_\beta \neq \emptyset$ then $X_\alpha = X_\beta$ for all α and β in Λ. The third condition means that the elements of P are **pair-wise disjoint**.

Theorem 2. *Let G be a group that acts on a set X via left multiplication. Furthermore, let P be the collection of G-orbits of X. The elements of P collectively partition X.*

Proof. Let G be a group that acts on a set X. Furthermore, let P be the collection of all G-orbits of X. We will show that the elements of P collectively partition X. A G-orbit of x must contain x since $e(x) = x$ for the identity $e \in G$. Hence every G-orbit is non-empty.

For every x and y in X, let $\mathcal{O}_x$ and $\mathcal{O}_y$ be the corresponding G-orbits respectively. Suppose that $\mathcal{O}_x \cap \mathcal{O}_y$ is non-empty. Therefore, there is a $z \in \mathcal{O}_x \cap \mathcal{O}_y$. Since $z \in \mathcal{O}_x$, by definition $g(x) = z$ and $g^{-1}(z) = x$ for some $g \in G$. Likewise, since $z \in \mathcal{O}_y$, it follows that $g'(y) = z$ and $g'^{-1}(z) = y$ for some $g' \in G$. But, G is closed under inverses and so both $g^{-1}g'$ is in G. We see that $g^{-1}g'(y) = x$ as well as $g'^{-1}g(x) = y$ which shows x and y are in the same orbit. For every $x' \in \mathcal{O}_x$ and $y' \in \mathcal{O}_y$, by definition, $x' = h(x)$ and $y' = h'(y)$ for some h and h' in G. Thus, we find that $h(x) = hg^{-1}g'(y)$ and $h'(y) = h'g'^{-1}g(x)$ and so $x' \in O_y$ and $y' \in O_x$. Thus by double inclusion, $\mathcal{O}_x = \mathcal{O}_y$. So we have shown that distinct G-orbits are disjoint.

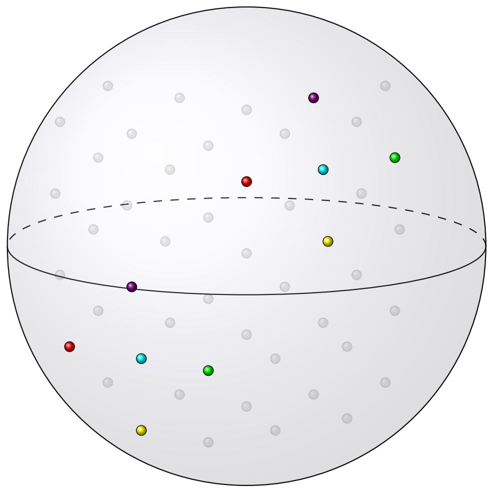
It suffices to show that $\bigcup_{x \in X} \mathcal{O}_x = X$. Every $x \in X$ is in the G-orbit of x and so $x \in \bigcup_{x \in X} \mathcal{O}_x$. Likewise, every $\mathcal{O}_x \subseteq X$ and so $\bigcup_{x \in X} \mathcal{O}_x \subseteq X$. By double inclusion, $\bigcup_{x \in X} \mathcal{O} = X$.

Since all criteria have been met, The G-orbits of X collectively partition X as desired.

[3] See Rudin's Principles of Analysis [Rud76] for more information regarding infinite sets of different cardinality.

Definition 11. Let G be a group that acts on a set X and suppose that $E \subseteq X$. A **coset** of E is the set $\{g(x)\colon x \in E\}$ for some $g \in G$ and is commonly denoted as gE.

Example 5. *In the following example, E contains many points, but we restrict our attention to the two cyan colored points. The cosets of E are also pairs of points and are color coded based on which coset they are in. Notice that a point is in gE if and only if it comes from $g(x)$ for some point $x \in E$.*



Theorem 3. *Let G be a group that acts on a set X via left multiplication without nontrivial fixed points. Suppose $R \subseteq X$ contains exactly one element from each distinct G-orbit and nothing more. Let P be the collection of all cosets of R. It follows that the elements of P collectively partition X.*

Proof. Let G be a group that acts on a set X via left multiplication without nontrivial fixed points. Suppose $R \subseteq X$ containing exactly one element from each distinct G-orbit and nothing more. Let P be the collection of all cosets of R. Furthermore, let gR and $g'R$ be arbitrary elements in P and suppose $gR \cap g'R$ is non-empty. Thus there is a $z \in gR \cap g'R$ and can be written as either gr or $g'r'$ for some r and r' in R. If $r \neq r'$, then z is in two different G-orbits by definition of R, which is impossible due to Theorem 2. Since, $gr = z = g'r$ and G acts on X without nontrivial fixed points, it follows that $g^{-1}g' = e$ and thus $g = g'$, which means $gR = g'R$. So the elements of P are pair-wise disjoint.

Next, we will show that the elements of P form all of X. For every $x \in X$, let $y \in \mathcal{O}_x \cap R$. By definition, $g(x) = y$ for some $g \in G$ and thus $x = g^{-1}(y)$, which means, $x \in g^{-1}(R)$. So X is a subset of the cosets of R. Every element of P is a subset of X and so the union of all elements of P must be a subset of X. By double inclusion, the elements of P collectively partition X as desired. This completes our proof.

It should be noted that G acting on X without nontrivial fixed points is crucial to this proof. Banach and Tarski's main contribution is how to get around this requirement. We are now ready to talk about the Banach-Tarski paradox.

The Paradox

Definitions and Theorem

We are interested in the process of applying group actions on subsets that partition a set A to form a new set B. Specifically, we are concerned with which sets are accessible to A via this transformation. Sometimes a group will act on a subset the same way as another subset. When this happens, it is helpful to talk about a collection of group elements that allow for repetition.

Definition 12. A **multi-set** is the same as a set, except that it does not ignore repetition. That is, a multi-set is allowed to have elements that are not distinct.

With this definition, we can properly define what we mean by transforming a set A into a new set B.

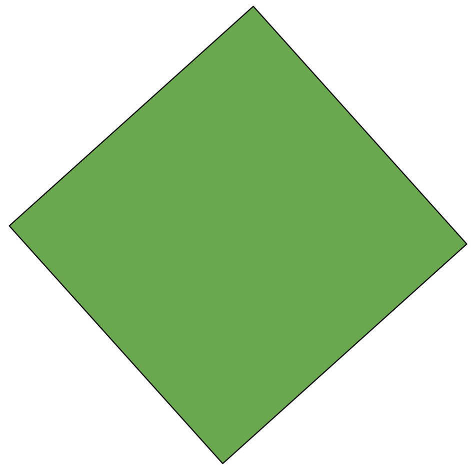
Definition 13. [4] Let G be a group that acts on a set X via left multiplication with subsets A and B. If there are two sets $\{A_i\}_{i=1}^k$ and $\{B_i\}_{i=1}^k$ whose elements collectively partition

[4] The definitions and theorems found in this section from now on exclusively come from Wagon's Banach-Tarski Paradox [Wag93].

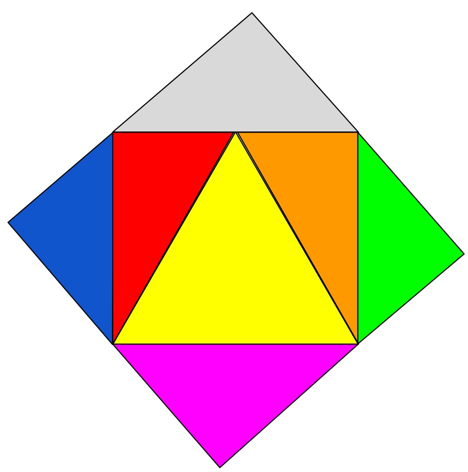
$A \subset X$ and $B \subset X$, respectively, and if there there is a multi-set, $\{g_i \in G\}_{i=1}^{k}$, such that $g_i(A_i) = B_i$ for all $i = 1 \ldots k$, then A is **finite G-equidecomposable** to B using k pieces.

Example 6. Consider the group SO_2 which is the group of all rotations in 2 dimensional space.

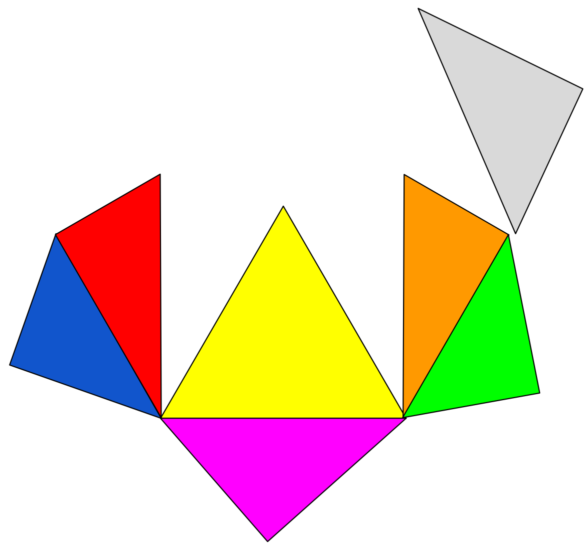
We start with a set called A.

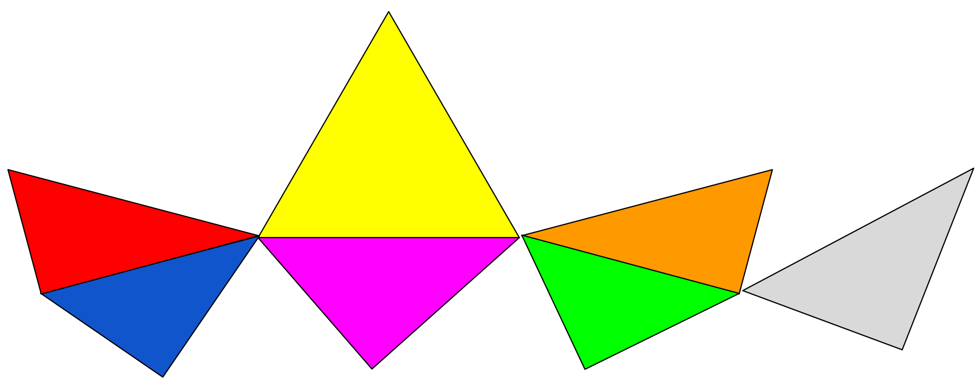


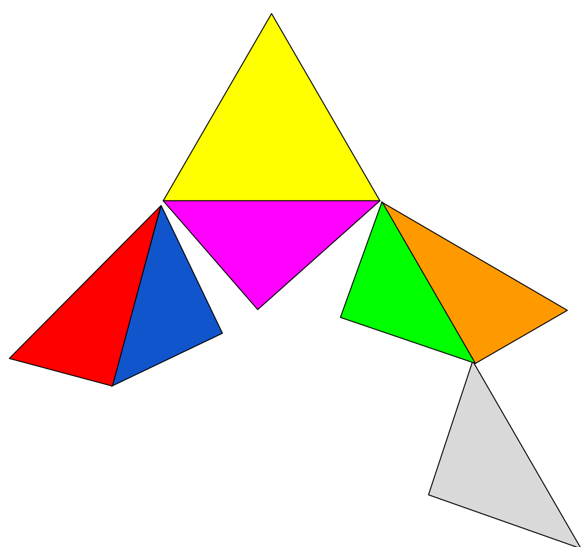
We collectively partition A.

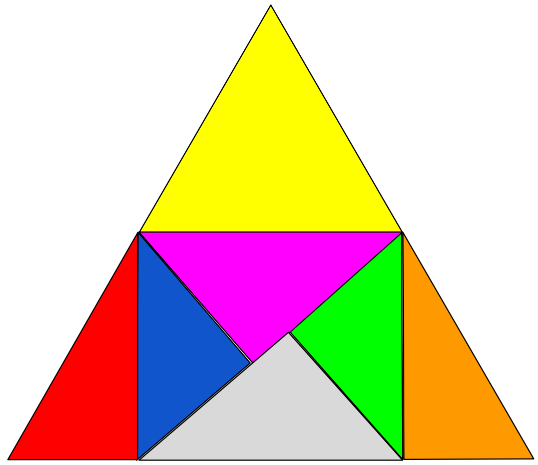


We rotate the pieces via SO_2 actions.

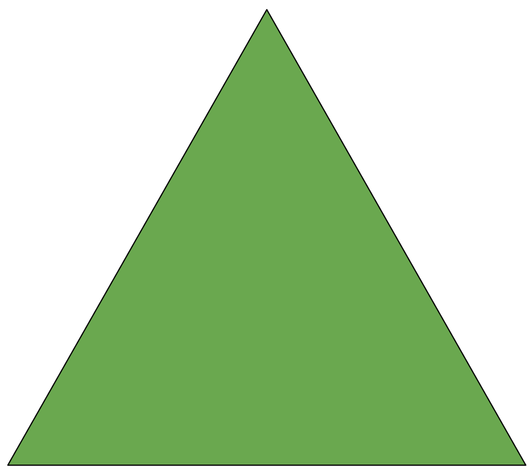








Finally, we union everything together.



We now have a set called B.

A diamond is finite SO_2-equidecomposable to an equilateral triangle.

In the case of polygons, the areas remain invariant under the group actions. When we consider scatterings of points as pieces, the areas may change.

Definition 14. Let G be a group that acts on a set X and suppose $E \subseteq X$. We say that E is **G-paradoxical** (or paradoxical with respect to G) if for some positive integers n and

m, there are two sets $\{A_i \subseteq E\}_{i=1}^{n}$ and $\{B_j \subseteq E\}_{j=1}^{m}$, whose elements, together, are pairwise disjoint, and there are two multi-sets $\{g_i \in G\}_{i=1}^{n}$ and $\{g'_j \in G\}_{j=1}^{m}$ such that $E = \bigcup_{i=1}^{n} g_i(A_i)$ and $E = \bigcup_{j=1}^{m} g'_j(B_j)$.

Example 7. Here we have a picture of a subset E that is G-paradoxical. Note that Rot is a rotation caused by the action of an element of G. Polygon pieces cannot be used to create a paradoxical decomposition; however, this picture is here to help make sense of the G-paradoxical definition. Usually the pieces would be scatterings of points instead.

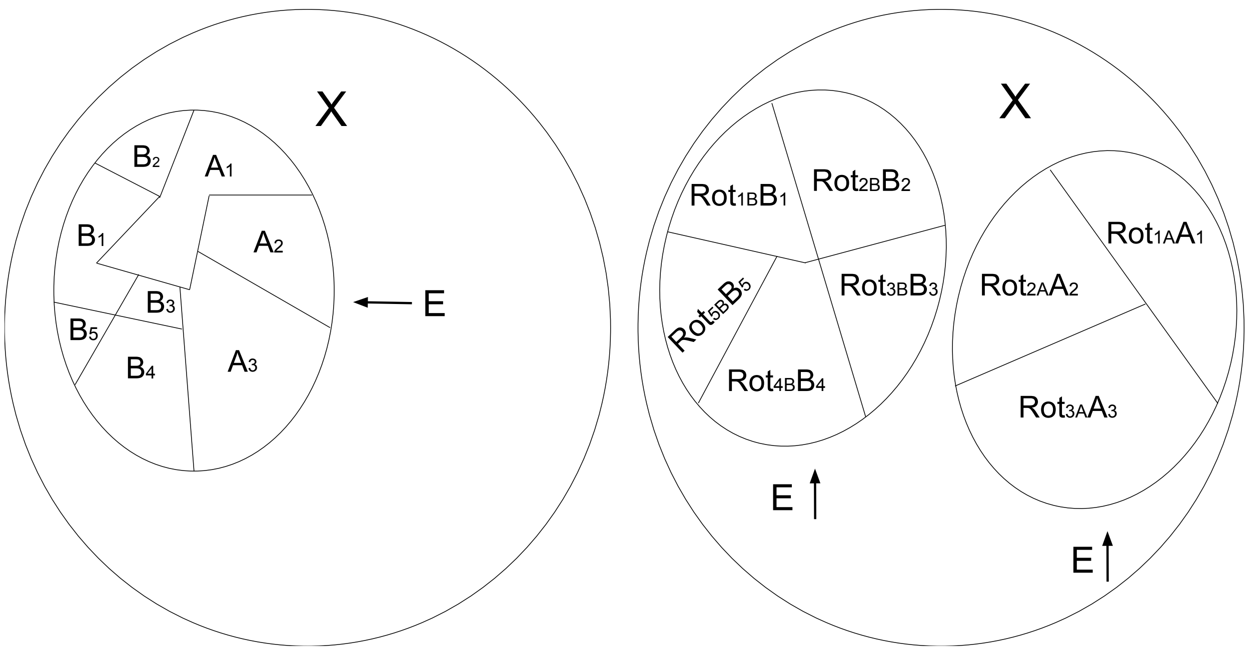


Figure 4: Depiction of a G-paradoxical set.

Notice that this definition does not depend on how G acts on X. In other words, the group action is arbitrary, but fixed. When a group is paradoxical with respect to itself, we call it a *paradoxical group*. Next, we show that the paradoxical property is shared amongst all sets that are finite G-equidecomposable to one another. This theorem is crucial when we transition from the Hausdorff paradox to the Banach-Tarksi paradox. We need a little lemma first.

Lemma 1. *Let G be a group that acts on a set X via left multiplication and suppose $E \subseteq X$*

is G-paradoxical. If $E = \bigcup_{i=1}^{n} g_i(A_i)$*, then there exists a set* $\{\mathcal{A}_i\}_{i=1}^{n}$ *such that the elements of* $\{g_i(\mathcal{A}_i)\}_{i=1}^{n}$ *are pairwise disjoint and the equation* $E = \bigcup_{i=1}^{n} g_i(\mathcal{A}_i)$ *holds.*

Proof. This is a proof by construction. Let G be a group that acts on a set X and suppose $E \subseteq X$ is G-paradoxical. This means there is a positive integer n such that the collection $\{A_i \subseteq E\}_{i=1}^{n}$ along with the multi-set $\{g_i \in G\}_{i=1}^{n}$ satisfies $E = \bigcup_{i=1}^{n} g_i(A_i)$. For each A_i, we start with A_1 and inducting up, if $g(A_i) \cap \bigcup_{j=i+1}^{n} g(A_j) \neq \emptyset$, then we reassign A_i to the set $\mathcal{A}_i$ defined by

$$g_i^{-1}\left[g_i(A_i) \setminus \left((g_i(A_i) \cap \bigcup_{j=i+1}^{n} g_j(A_j)]\right)\right].$$

Once $A_i = A_n$ we finish without reassigning A_i. Notice that $\bigcup_{i=1}^{n} g_i(A_i) = \bigcup i = 1^n g_i(\mathcal{A}_i)$. Furthermore, every $g(\mathcal{A}_i)$ will be pairwise disjoint as desired.

This lemma addresses the possibility that the sets $g_i(A_i)$ may overlap each other in some ways, so the algorithm carves out the redundant points in the A_i sets. Due to this lemma, if a set E is G-paradoxical, then the elements of $\{g_i(\mathcal{A}_i)\}_{i=1}^{n}$ form a partition of E.

Notation. Whenever a set E is G-paradoxical, we will only consider collections of subsets whose elements, once relevantly acted on by G, collectively partition E.

The sphere cannot be duplicated directly. However, the unit sphere is SO_3-equidecomposable to a set that can be duplicated directly, which we will prove later. This next theorem is necessary to complete the proof of the Banach-Tarski Paradox.

Theorem 4. *Let* G *be a group that acts on* X *and suppose* E *and* E' *are finite G-equidecomposable subsets of* X *using* k *pieces. If* E *is G-paradoxical, then so is* E'*.*

Proof. Let G be a group that acts on X and suppose E and E' are G-equidecomposable subsets of X with E being G-paradoxical. Thus, there exists two positive integers n and

m, such that the collections $\{A_i \subseteq E\}_{i=1}^{n}$ and $\{B_j \subseteq E\}_{j=1}^{m}$, whose elements are collectively pair-wise disjoint, can be found along with two multi-sets $\{g_i \in G\}_{i=1}^{n}$ and $\{g'_j \in G\}_{j=1}^{m}$ such that $E = \bigcup_{i=1}^{n} g_i(A_i)$ and $E = \bigcup_{j=1}^{n} g'_j(B_j)$. Since E' is finite G-equidecomposable to E using k pieces, there are two sets $\{E_i\}_{i=1}^{k}$ and $\{E'_i\}_{i=1}^{k}$ whose elements collectively partition $E \subset X$ and $E' \subset X$ respectively along with a multi-set, $\{h_l \in G\}_{l=1}^{k}$, such that $h_l(E_l) = E'_l$ for all $l = 1 \ldots k$. Consider the collections of intersections between the different partitions of E. That is, let I_A and I_B be defined by

$$\{g_i(A_i) \cap E_l \mid g_i(A_i) \cap E_l \neq \emptyset \text{ for all } i = 1 \ldots n \text{ and for all } l = 1 \ldots k\}$$

and

$$\{g'_j(B_j) \cap E_l \mid g'_j(B_j) \cap E_l \neq \emptyset \text{ for all } j = 1 \ldots n \text{ and for all } l = 1 \ldots k\}$$

respectively. The above two sets are individually pairwise disjoint because $\{g_i(A_i)\}_{i=1}^{n}$, $\{g'_j(B_j)\}_{j=1}^{m}$ and $\{E_l\}_{l=1}^{k}$ are collectively pairwise disjoint. It follows directly that

$$\bigcup_{l=1}^{k}\bigcup_{i=1}^{n}(h_l(g_i(A_i) \cap E_l)) = E'$$

and

$$\bigcup_{l=1}^{k}\bigcup_{j=1}^{m}\big(h_l(g'_j(B_j) \cap E_l)\big) = E'$$

and so E' is G-paradoxical as desired.

Next, we will show that a certain subset of the unit sphere is SO_3–paradoxical. In order to do this, we need a paradoxical group to act on it. First, we will prove that certain free groups are paradoxical.

Theorem 5. *Any free group of rank 2 is a paradoxical group.*

Proof. Let F be any free group, with identity e, generating by the set X defined as $\{a, b\}$.

Furthermore, let F act on itself by left multiplication. We will show that F is paradoxical with respect to itself. We partition F by collecting all words that begin with the same letter. Let $W_a, W_b, W_{a^{-1}}$, and $W_{b^{-1}}$ each be the set of all reduced words that start on the left with the letter indicated by W's subscript. Since no word can start on the left with more than one letter, it follows that $W_a, W_b, W_{a^{-1}}$, and $W_{b^{-1}}$ are pair-wise disjoint. Every word in F either is e, or is a word that starts on the left exclusively with either the letters a, b, a^{-1} or b^{-1}. So $F = W_a \cup W_b \cup W_{a^{-1}} \cup W_{b^{-1}} \cup \{e\}$ and so these subsets collectively partition F. Let us choose the collections of subsets $A = \{W_a, W_{a^{-1}}\}$ and $B = \{W_b, W_{b^{-1}}\}$ along with the multi-sets $\{a^{-1}, e\}$ and $\{b^{-1}, e\}$. Since every word in W_a is reduced, the second letter of each word can be any letter other than a^{-1} by definition of F. If the word in W_a is a itself, then a becomes e. Hence, $a^{-1}W_a = W_a \cup W_b \cup W_{b^{-1}} \cup \{e\}$ and similarly, $b^{-1}W_b = W_a \cup W_b \cup W_{a^{-1}} \cup \{e\}$. These equations imply $a^{-1}W_a \cup W_{a^{-1}} = F$ and $b^{-1}W_b \cup W_{b^{-1}} = F$. By definition, F is paradoxical with respect to itself as desired.

Now that we know one group is paradoxical, we will want to find ways of using this group. In order to do this, we need the Axiom of Choice, which we address next.

Definition 15. Let X be a collection of subsets from a given universal set. A **choice function** of X is a function f which associates to each nonempty element E of X an element of E.

Example 8. Let U be the universal set defined by $U = \{a, b, c, d, e\}$. Furthermore, let X be the collection $\{E_1, E_2, E_3\}$ such that $E_1 = \{a, b, c\}$, $E_2 = \{a, b, c, d\}$, $E_3 = \{a, b, c, d, e\}$. One such choice function is f defined by $f\colon X \setminus \emptyset \to \bigcup_{i=1}^{3} E_i$ such that $f(E_1) = a$, $f(E_2) = b$ and $f(E_3) = e$. In this scenario, there can be many choice functions; specifically, one for each distinct set of assignments.

The Axiom of Choice. Every collection of sets has a choice function.

Although, this axiom seems intuitive and obvious, it has become quite controversial. This is especially true when considering a collection with infinite cardinality. The Axiom of choice allows for a correspondence between subsets of X and subsets of the free group F acting on X. This next theorem gives precision to this idea.

Theorem 6. *If G is a paradoxical group that acts on a set X, without nontrivial fixed points, then X is G-paradoxical.*

Proof. Let G be a paradoxical group that acts on a set X and without nontrivial fixed points. Thus, there exist two positive integers n and m, two collections $\{A_i \subseteq G\}_{i=1}^{n}$ and $\{B_j \subseteq G\}_{j=1}^{m}$, whose elements are pairwise disjoint, and two multi-sets $\{g_i \in G\}_{i=1}^{n}$ and $\{g'_j \in G\}_{j=1}^{m}$, so that $G = \bigcup\limits_{i=1}^{n} g_i(A_i)$ and $G = \bigcup\limits_{j=1}^{n} g'_j(B_j)$. By the Axiom of Choice, there is a choice function f on the collection of all G-orbits. So let $R \subseteq X$ be the set $\{f(\mathcal{O}_x)\colon x \in X$ Consider the the collection of all cosets of R defined by $\{g(R)\colon g \in G\}$. Due to Theorem 3, the cosets of R partition X. For each A_i and B_j we define A_i^* and B_j^* by $A_i^* = \left\{\bigcup\limits_{g \in A_i} g(R)\right\}$ and $B_j^* = \left\{\bigcup\limits_{g \in B_j} g(R)\right\}$ respectively. Since $G = \bigcup\limits_{i=1}^{n} g_i(A_i)$ and $G = \bigcup\limits_{j=1}^{m} g'_j(B_j)$, it follows directly that $X = \bigcup\limits_{i=1}^{n} g_i(A_i(R)) = \bigcup\limits_{i=1}^{n} g_i(A_i^*)$ and similarly, $X = \bigcup\limits_{j=1}^{m} g'_j(B_j^*)$.

I like to call this theorem the "spread" of paradox. In essence, we are taking the paradoxical property of groups and applying it to sets acted on. An important point to observe is that we can spread the paradox property in this way only if we can use the Axiom of Choice. Without the Axiom of Choice, there may not exist a set whose cosets cover the entire sphere. The Hausdorff Paradox is a specific instance of spreading paradox. Unlike the Banach-Tarski paradox, the Hausdorff Paradox requires taking out all the nontrivial fixed points on the sphere in order to effectively use this theorem.

Theorem 7 (Hausdorff Paradox)**.** *There is a countable subset D of S^2 such that $S^2 \setminus D$ is SO_3-paradoxical.*

Proof. Due to Theorem 1 there exist two elements ϕ and θ in SO_3 that generate a free subgroup, F, of rank 2. Given any reduced word w in ϕ and θ, there exist two nontrivial fixed points in S^2 located where the axis of rotation intersects S^2. Let D be the set of all nontrivial fixed points of S^2 caused by F. Since there are a countable number of reduced words in ϕ and θ, there are a countable number of such fixed points. By definition of D, F acts on $S^2 \setminus D$ without nontrivial fixed points. We know that this free subgroup is itself paradoxical, due to Theorem 5. So it follows that $S^2 \setminus D$ is F-paradoxical due to Theorem 6. Since the two multi-sets of elements of F that make $S^2 \setminus D$ F-paradoxical are also multi-sets of elements of SO_3, it follows that $S^2 \setminus D$ is SO_3-paradoxical as desired.

Notation. For the rest of this section, SO_3 will be acting on S^2 via matrix-vector multiplication. Whenever F is mentioned, we mean the free subgroup of rank 2 contained in SO_3 with generating set $\{\phi, \theta\}$. Moreover, when D is mentioned, we mean the countable subset of nontrivial fixed points of S^2 caused by F.

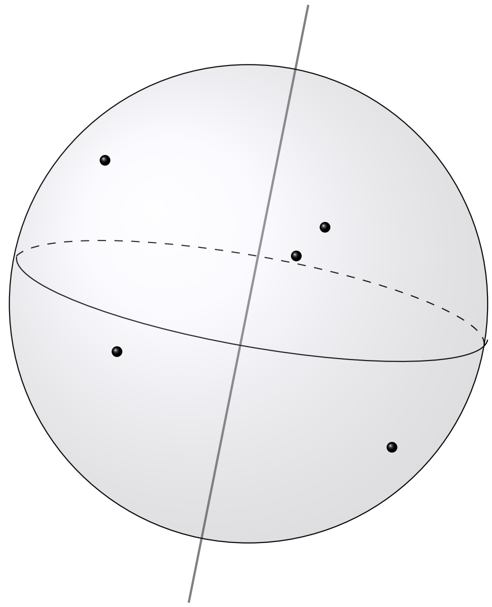
Now, we are ready to show Banach and Tarski's contributions to generalize Hausdorff's paradox. Specifically, Banach and Tarski found a way to ignore the countable subset D.

Theorem 8. *There exists a line, ℓ, that contains the origin in $\mathbb{R}^3$ and intersects S^2 at two points not found in D.*

Proof. Consider the set $S^2 \setminus D$. Since S^2 has an uncountable number of points and D has a countable number of points, there exists a point $x \in S^2 \setminus D$. Since $x \in S^2 \setminus D$, $-x \in S^2 \setminus D$ as well. Let ℓ be the line through x and the origin. By definition, ℓ intersects S^2 at two points not found in D as desired.

Notation. Whenever ℓ is mentioned in this section, we mean the line above.

Example 9. Here is what the situation looks like.

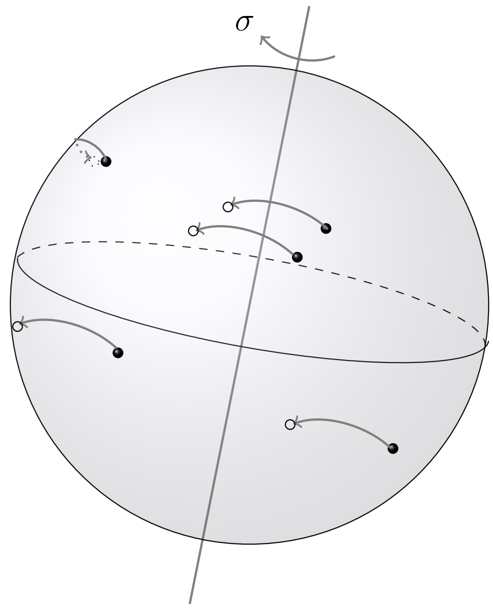


Note that the black points represent the points in D and the line is ℓ. Not every point in D can be drawn but the picture should give the general idea.

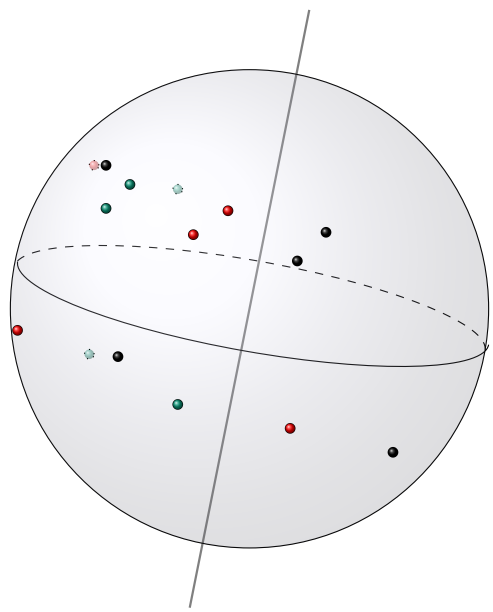
Theorem 9. *Suppose there is a rotation σ about the line ℓ such that the elements of the collection $\{\sigma^{i-1}D \colon i \in \mathbb{N}\}$ are pairwise-disjoint. It follows that $S^2 \setminus D$ is SO_3-equidecomposable to S^2 using two pieces.*

Proof. Suppose there is a rotation σ about the line ℓ such that the elements of $\{\sigma^{i-1}(D) \colon i \in \mathbb{N}\}$ are pairwise disjoint. Let $\overline{D}$ be the set defined by $\bigcup_{n=0}^{\infty} \sigma^n(D)$. Observe that $S^2 = \overline{D} \cup S^2 \setminus \overline{D}$. If we apply $\sigma \in SO_3$ to $\overline{D}$ and the do identity to $S^2 \setminus \overline{D}$, we find that $\overline{D} \cup S^2 \setminus \overline{D}$ is SO_3-equidecomposable to $\sigma(\overline{D}) \cup S^2 \setminus \overline{D}$ using two pieces. However, $\sigma(\overline{D})$ is just $\overline{D} \setminus D$ and so $\sigma(\overline{D}) \cup S^2 \setminus \overline{D} = S^2 \setminus D$ as desired.

Example 10. Given the previous example, we apply a rotation σ about the line ℓ to every point in D like so:



This leads us to obtaining new points $\sigma(D)$. We do this process again. Notice that black points are in D, red points are in $\sigma(D)$ and green points are in $\sigma^2(D)$.



We union all the sets $\sigma^n(D)$ for all $n \in \mathbb{N}$ to create the set $\overline{D}$. If we take all of $\overline{D}$ and rotate it again to get $\sigma(\overline{D})$, then the points in D will transform into the points in $\sigma(D)$ and the points in $\sigma(D)$ will transform into the points in $\sigma^2(D)$ etc. What this means is that $\sigma(\overline{D})$ is really just $\overline{D}$ without any points in D.

In order to take full advantage of Theorem 9, we need to find a suitable angle of rotation around the line ℓ that works. Specifically, we need for every point x on the unit sphere, $x \in \sigma^n(D)$ and $x \in \sigma^m(D)$ if and only if $n = m$. To do this, we need to show that there only is a countable number of angles that will not work.

Theorem 10. *Let A be the set of rotations in SO_3 around ℓ defined by*

$$A = \{\alpha \mid \textit{There is a } P \in D \textit{ and an } n \in \mathbb{N} \textit{ such that } n\alpha(P) \in D\}.$$

It follows that A is a countable set.

Proof. Let A be the set of all rotations in SO_3 around ℓ defined by

$$A = \{\alpha \mid \text{There is a } P \in D \text{ and an } n \in \mathbb{N} \text{ such that } n\alpha(P) \in D\}.$$

For each $P \in S^2$, let A_P be the set of rotations in SO_3 around ℓ defined by $\{\alpha \mid \alpha(P) \in D\}$. Finally, let A' be the set defined by $\bigcup_{P \in D} \bigcup_{\alpha \in A_P} \{\frac{\alpha}{n} \mid n \in \mathbb{N}\}$. Since D is countable, A_P is countable as well. Since countable unions of countable sets are countable, it follows that A' is countable. So it suffices to show that $A = A'$.

Let $\gamma \in A'$ be arbitrary. Thus for some point P, there is an $\eta \in A_P$ with $\gamma = \frac{\eta}{n}$ for some $n \in \mathbb{N}$. Since $\eta \in A_P$, we see that $\eta(P) \in D$ and so $n\gamma(P) \in D$ which means γ is in A.

Next, let $\omega \in A$ be arbitrary. So there is a $P \in D$ and an $n \in \mathbb{N}$ such that $n\omega(P) \in D$. By definition, $n\omega \in A_P$ and so $\frac{n\omega}{n} \in A'$ which means $\omega \in A'$. By double inclusion, $A = A'$ as desired.

Theorem 11. *There exists a rotation σ in SO_3 about ℓ such that the elements of $\{\sigma^{i-1}(D) \colon i \in \mathbb{N}\}$ are pairwise disjoint.*

Proof. Let A be the set of angles α such that for some $n > 0$ and some $P \in D$, $n\alpha(P)$

is also in D. Due to Theorem 10, A is a countable set. However, there is an uncountable number of angles that rotate around ℓ and so we can pick an angle σ that isn't in A. Since σ isn't in A, it follows by definition that $\sigma^n(D) \cap D = \emptyset$ for all $n > 0$. Let us write n of the form $s - t$ where $s > t \geq 0$. If we apply σ^t to both sides of $\sigma^n(D) \cap D = \emptyset$, we arrive with $\sigma^s(D) \cap \sigma^t(D) = \emptyset$ for all $s > t \geq 0$. Hence, we have shown that the elements of $\{\sigma^{i-1}(D) \colon i \in \mathbb{N}\}$ are pairwise disjoint as desired.

The Banach-Tarski Paradox. *S^2 is SO_3-paradoxical.*

Proof. By the Hausdorff paradox, $S^2 \setminus D$ is paradoxical with respect to SO_3. From Theorem 9, we know that $S^2 \setminus D$ is finite SO_3-equidecomposable to S^2 using 2 pieces. Since $S^2 \setminus D$ is paradoxical with respect to SO_3 and $S^2 \setminus D$ is finite SO_3-equidecomposable to S^2 using 2 pieces, it follows from Theorem 4 that S^2 is paradoxical with respect to SO_3 as desired. This completes our proof.

Conclusion

The Banach-Tarski paradox is a surprising consequence of including the Axiom of Choice when constructing Modern Set Theory. So why is the Axiom of Choice still used in mathematics? The answer is quite surprising too. That is, the paradoxes may be even worse if we assume the Axiom of Choice is not true. As several community members of Mathoverflow [hdh] point out, without the Axiom of Choice, the following bizarre consequences are true:

1. There can be a nonempty tree, with no leaves, but which has no infinite path. That is, every finite path in the tree can be extended one more step, but there is no path that goes forever.

2. A real number can be in the closure of a set $X \subset \mathbb{R}$, but not the limit of any sequence from X.

3. A function $f : \mathbb{R} \to \mathbb{R}$ can be continuous in the sense that $x_n \to x \implies f(x_n) \to f(x)$, but not in the ϵ δ sense.

4. A set can be infinite, but have no countably infinite subset. Thus, it can be incorrect to say that $\aleph_0$ is the smallest infinite cardinality, since there can be infinite sets of incomparable size with $\aleph_0$.

5. There can be an equivalence relation on $\mathbb{R}$, such that the number of equivalence classes is strictly greater than the size of $\mathbb{R}$.

6. There can be a field with no algebraic closure.

7. The rational field $\mathbb{Q}$ can have different non-isomorphic algebraic closures

8. There can be a vector space with no basis.

9. There can be a vector space with bases of different cardinalities.

10. The reals can be a countable union of countable sets. Consequently, the theory of Lebesgue Measure can fail totally.

11. The first uncountable ordinal ω_1 can be singular. More generally, it can be that every uncountable $\aleph_\alpha$ is singular. Hence, there are no infinite regular uncountable well-ordered cardinals.

For these reasons, mathematicians accept the Axiom of Choice and the Banach-Tarski Paradox. To reconcile this paradox, mathematicians have developed Measure Theory which helps resolve the problems by addressing the volume of these spheres. Furthermore, the Banach-Tarski paradox has been used to model hadron physics. Specifically, when a quark is pulled out from mesons or bosons, the strong nuclear force is so strong, that the quark splits and the original particle returns and we are left with a virtual particle.

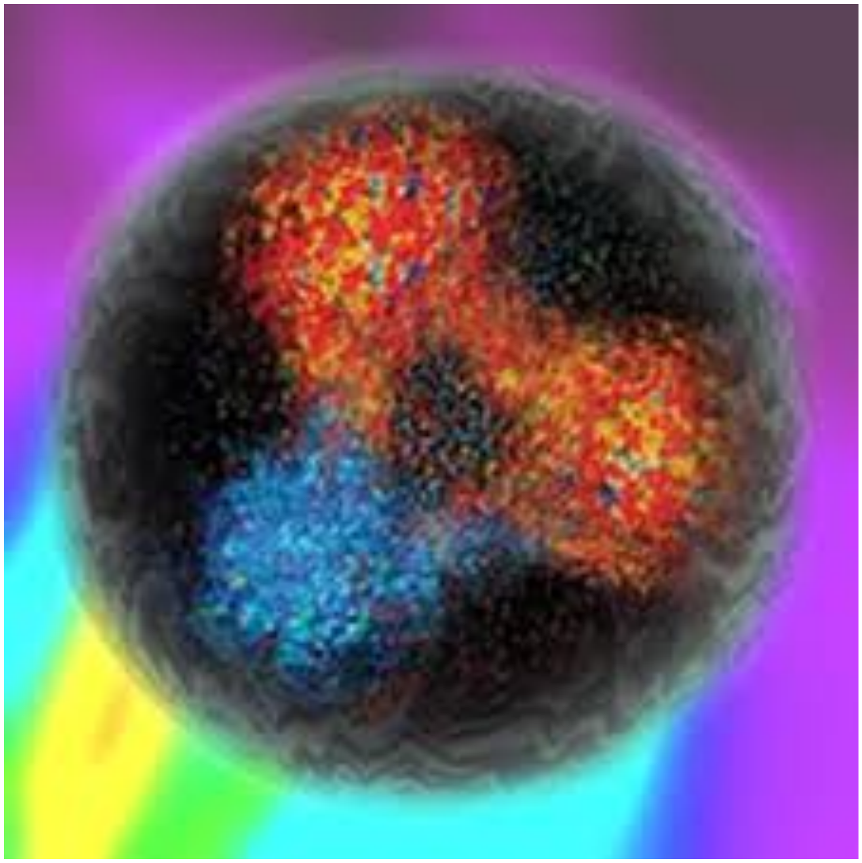


Figure 5: The quark hadron of a proton.

More can be found from B. W. Augenstein's paper on Hadron Physics and Transfinite Set Theory [Aug84].

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Taking the five loaves... he gave thanks and broke the loaves... and the disciples picked up twelve basketfuls of broken pieces that were left over. The number of those who ate was about five thousand...

Matthew on Jesus 14: 19-21