The Gamma Functio**n!**

A Senior Mathematics Essay

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Chapter 1

An Introduction to the Gamma Function

This paper is an examination of the product of one mathematician's curiosity, *The Gamma Function*. The aim of this discussion is to provide the reader with an understanding of this function, examine some of its most notable properties, and take a closer look at several of its applications.

In the words of an American mathematician, *"Each generation has found something of interest to say about the gamma function. Perhaps the next generation will also."*[1] It is my hope to inspire curiosity and further thought into the Gamma Function.

Before we can discuss the Gamma Function, however, it is only appropriate for us to first introduce the function that precedes the Gamma Function in a very significant way, serving as its sole inspiration.

[1]Philip J. Davis (American Mathematician), (1963)

L. Arbogast

1.1 The Factorial Function

We give a brief definition of the factorial function, a function that is of tremendous importance to the field of combinatorics. n **factorial** (or $n!$) is the product of all positive integers less than or equal to n.

$$\begin{aligned}
1! &= 1 \\
2! &= 2 \cdot 1 \\
3! &= 3 \cdot 2 \cdot 1 \\
&\vdots \\
n! &= n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1
\end{aligned}$$

By convention, we have that $0! = 1$, since $0!$ represents an empty product as there are no nonegative integers less than or equal to 0.

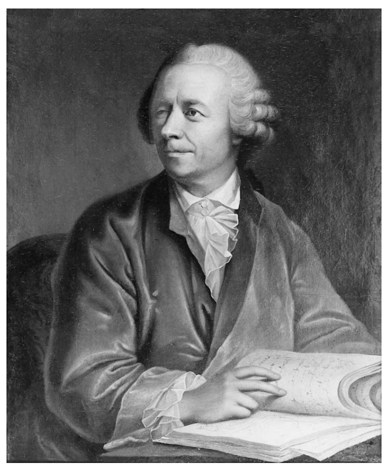
Louis François Antoine Arbogast is the 18th century French Mathematician who is credited for introducing the concept of *factorial* as we have just defined it.

1.2 The Solution to a Problem

It should be noted that several fields of mathematics rely heavily on our ability to understand and manipulate the factorial function, but the function itself is defined to be discrete, taking on only nonnegative-integer values.

Interestingly, the development of the Gamma function came into being as the solution to an intriguing proposal:

Can the factorial function be extended beyond integers? [2]



L. Euler

In 1720[3], the Swiss mathematician, Leonard Euler, introduced the world to his *Gamma Function*, as a solution to the problem of extending the factorial function to the real and complex number systems.

From this point forward, the Gamma function will be the primary focus of our discussion.

[2]This problem was supposedly proposed by mathematician Bernoulli

[3]The Gamma function was developed through an exchange of letters between other great mathematicians of this era.

1.3 Definitions

Students of undergraduate mathematics might agree that a picture is worth a thousand words. In a discipline where concepts seem to grow increasingly complex and abstract (no pun intended), one can only hope for a definition that can be understood without tackling an arsenal of precursory theorems. Fortunately, our first impression of the Gamma function fits this profile.

The Gamma Function: $\Gamma(p)$

The **Gamma function** can first be defined for $p > 0$ as,

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx$$

Shortly we will explain exactly what is meant by the term *well defined.* Below, we see graph of this function for $p > 0$,

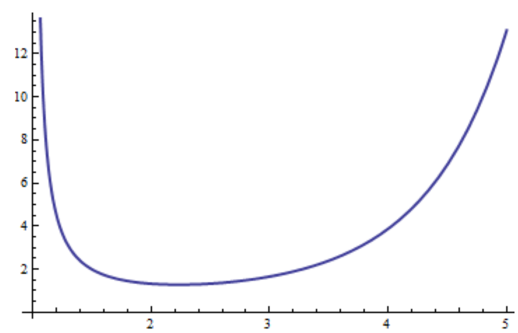


Figure 1.1

It should be noted that this particular function is defined, and has applications, in the Real and Complex Number systems (which will be denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively).

This essay intends to prepare readers with the analytic tools necessary for understanding the properties of the real-valued Gamma function, then discuss the function's importance in radically differing contexts.

Improper Integral

Consider that the Gamma Function,

$$\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}dx$$

is not an integral evaluated over a finite interval. As a result of this, we must define a way to classify this type of integral.

An integral of the form

$$\int_a^\infty f(x)dx$$

for some $a \in \mathbb{R}$, is called an **improper integral** because the Riemann Integral requires the interval of integration to be an interval of finite length.

Such an improper integral is defined as follows,

$$\int_a^\infty f(x)dx = \lim_{C\to\infty} \int_a^C f(x)dx$$

Hence, an improper integral exists (and is called *convergent*) when the limit exists.

A Well Defined Function

The Gamma function is said to be **well defined** for $p \in (0, \infty)$ because the $\Gamma(p)$ uniquely assigns an output for any value of p in this interval.

We claim the following to be true:

The Gamma function,

$$\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}dx$$

is well defined for any p on the interval $(0, \infty)$; also note that $x \in (0, \infty)$

To show that the Gamma function is well defined we must show that the integral is convergent. To do so, it is convenient to express the Gamma Function in the following way:

$$\int_0^\infty x^{p-1}e^{-x}dx = \int_0^1 x^{p-1}e^{-x}dx + \int_1^\infty x^{p-1}e^{-x}dx \tag{1.1}$$

We are determined to show that each of the integrals on the right side of (1.1) is convergent. To do so, we must first prove a few supporting lemmas.

Lemma A

$\int_0^1 x^{p-1}e^{-x}dx$ is an absolutely convergent integral for $p > 0$.

Proof. First consider,

$$\int_0^1 x^{p-1}e^{-x}dx \tag{1.2}$$

when $x \in [0, 1]$, we have that $e^{-x} \leq 1$, and so $x^{p-1}e^{-x} \leq x^{p-1}$.

Integrating the right function, we have

$$\int x^{p-1}dx = \frac{1}{p}x^p + C$$

Case 1: $p \geq 1$

$$\int_0^1 x^{p-1}dx = \frac{1}{p}x^p\Big|_0^1 = \frac{1}{p} < \infty$$

Hence, the integral is absolutely convergent for $p \geq 1$.

Case 2: $p \in (0, 1)$

Now we are describing another type of improper integral, since for $p \in (0, 1)$, the function x^{p-1} is not defined at $x = 0$. So we must evaluate as follows,

$$\int_0^1 x^{p-1}dx = \lim_{C\to 0^+}\int_C^1 x^{p-1}dx = \lim_{C\to 0^+}\frac{1}{p}x^p\Big|_C^1$$

$$= \frac{1}{p} \lim_{C \to 0^+} (1 - C^p) = \frac{1}{p} < \infty$$

Thus, this integral is also absolutely convergent for $p \in (0, 1)$.

Since $x \in [0,1]$, $0 < \int_0^1 x^{p-1} e^{-x} dx \leq \int_0^1 x^{p-1} dx < \infty$, so we see that (1.2) is indeed absolutely convergent. $\square$

Next, consider the second integral where $x \geq 1$

$$\int_1^\infty x^{p-1} e^{-x} dx \tag{1.3}$$

Lemma B

$\int_1^\infty x^{p-1} e^{-x} dx$ is an absolutely convergent integral for $p > 0$.

Before we can show this, we'll need to prove two more lemmas (C & D).

Lemma C

For all $s > 0$, $\int_1^\infty e^{-sx} dx$ converges.

Proof. For $x > 0$

$$\begin{aligned}
\int_1^\infty e^{-sx} dx &= \lim_{p \to \infty} \int_1^\infty e^{-sx} dx \\
&= \lim_{p \to \infty} \left[-\frac{1}{s} e^{-sx} \right]_1^p \\
&= \lim_{p \to \infty} \left(-\frac{1}{s} \left[e^{-sp} - e^{-s} \right] \right) \\
&= -\frac{1}{s} [0 - e^{-s}] \\
&= \frac{1}{s} e^{-s} < \infty
\end{aligned}$$

Thus, $\int_1^{\infty} e^{-sx} dx$ converges. $\square$

Lemma D

Let $n \in \mathbb{Z}$, then

$$\lim_{x\to\infty} \frac{x^{n-1}}{e^{\frac{1}{2}x}} = 0$$

Proof. Note, showing this for $n \leq 1$, we may express the x^{n-1} as x^{-m}, for some negative integer $-m$.

We may rewrite this limit as

$$\lim_{x\to\infty} \frac{x^{-m}}{e^{\frac{1}{2}x}} = \lim_{x\to\infty} \frac{1}{e^{\frac{1}{2}x} x^m} = 0$$

Next, consider that for fixed $n > 1$, the limit of $\frac{x^{n-1}}{e^{\frac{1}{2}x}}$ as $x \to \infty$, gives the indeterminate form $\frac{\infty}{\infty}$.

As a result, we must use L'Hopitals rule to obtain

$$\lim_{x\to\infty} \frac{x^{n-1}}{e^{\frac{1}{2}x}} = \lim_{x\to\infty} \frac{(n-1)x^{n-2}}{\frac{1}{2}e^{\frac{1}{2}x}}$$

which again gives us the indeterminate form $\frac{\infty}{\infty}$.

Applying L'Hopitals a second time gives us

$$\lim_{x\to\infty} \frac{x^{n-1}}{e^{\frac{1}{2}x}} = \lim_{x\to\infty} \frac{(n-1)(n-2)x^{n-3}}{\frac{1}{2^2}e^{\frac{1}{2}x}}$$

which also gives us the indeterminate form $\frac{\infty}{\infty}$.

It is clear that applying this rule i times, for $i < n$, will give us a similar indeterminate form.

Applying L'Hopitals rule for the nth time gives

$$\lim_{x\to\infty}\frac{x^{n-1}}{e^{\frac{1}{2}x}}=\lim_{x\to\infty}\frac{(n-1)!x^{n-n}}{\frac{1}{2^n}e^{\frac{1}{2}x}}=\lim_{x\to\infty}\frac{(n-1)!}{\frac{1}{2^n}e^{\frac{1}{2}x}}=0$$

□

Now we have the tools to show that Lemma B is true. Recall that Lemma B states that $\int_1^\infty x^{p-1}e^{-x}dx$ is an absolutely convergent integral for $p>0$.

Proof. Consider the integral $\int_1^\infty x^{p-1}e^{-x}dx$, with $p>0$.
By Lemma D, for $\epsilon=1$, there exists some $M>0$ such that for all $x\geq M$,

$$|\frac{x^{p-1}}{e^{\frac{1}{2}x}}|<1$$

So, for $x\geq M$, $0\leq x^{n-1}<e^{\frac{1}{2}x}$.

Thus for all $x\geq M$,

$$0\leq e^{-x}x^{n-1}<e^{-x}e^{\frac{1}{2}x}=e^{-\frac{1}{2}x}$$

Notice that, by Lemma C with $s=\frac{1}{2}$, $\int_1^\infty e^{-\frac{1}{2}x}dx$ converges.

So by the comparison test,

$$\int_1^\infty x^{n-1}e^{-x}dx$$

is convergent for all $n\in\mathbb{Z}$.

Now for $p\geq 1$, let m be the greatest integer such that $m\leq p<m+1$, then for $x>0$,

$$0\leq e^{-x}x^{p-1}\leq e^{-x}x^m$$

Since we already know that $\int_1^\infty e^{-x}x^m dx$ is convergent, it follows that $\int_1^\infty e^{-x}x^{p-1}dx$ is also convergent.

For $p \in (0,1)$ and $x \in [1,+\infty)$, we know $x^{p-1} \leq x$.

Thus,

$$\frac{1}{e^{\frac{1}{2}x}} \leq \frac{x^{p-1}}{e^{\frac{1}{2}x}} \leq \frac{x}{e^{\frac{1}{2}x}}$$

Also, by Lemma B, $\lim_{x\to\infty} \frac{x}{e^{\frac{1}{2}x}} = 0$ and $\lim_{x\to\infty} \frac{1}{e^{\frac{1}{2}x}} = 0$.

Then, by the sandwich theorem, $\lim_{x\to\infty} \frac{x^{p-1}}{e^{\frac{1}{2}x}} = 0$.

So for $p \in (0,1)$, we have that

$$0 \leq x^{p-1}e^{-x} \leq e^{-\frac{1}{2}x}$$

which implies that

$$\int_1^\infty x^{p-1}e^{-x}dx \leq \int_1^\infty e^{-\frac{1}{2}x}dx < \infty$$

Therefore $\int_1^\infty x^{p-1}e^{-x}dx$ is convergent by the comparison test. □

Theorem 1.0: The Gamma Function is Well Defined

Proof. Since we've shown that the two integrals (1.2) and (1.3) are convergent, we see that the sum of these integrals is convergent.

Since the Gamma Function is equal to the sum of these integrals by equation (1.1), it is convergent for all $p \in (0,\infty)$.

By definition, the Gamma Function is well defined for all $p > 0$. □

Now that we have shown the Gamma Function to be well-defined, we may begin to discuss some of the properties it possesses. Such properties will be of interest to the applications mentioned later.

Chapter 2

The Gamma Function & Its Properties

2.1 The Factorial Function!

One of the most famous characteristics of the Gamma function, is that for all positive integers n, the values of $\Gamma(n+1)$ coincide with the values of the factorial function, $n!$.

The statement $\Gamma(p+1) = p\Gamma(p)$ is defined as the *recursive relation* for the Gamma function and holds for all $p \geq 0$.

Lemma E: The Recursive Relation for $\Gamma(p)$

For all $p > 0$,

$$\Gamma(p+1) = p\Gamma(p)$$

Proof. Consider the Gamma function for $p > 0$,

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx$$

Case 1: Consider when $p > 1$.

Recall that the Gamma Function is an improper integral, since is not defined at 0, so we must consider the *limit* of this integral as we approach zero from the left. we do so in the following way,

$$\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}dx = \lim_{C\to 0^+}\int_C^\infty x^{p-1}e^{-x}dx$$

Now, we pick u and dv such that,

$$\begin{bmatrix} u = e^{-x} & dv = x^{p-1}dx \\ du = -e^{-x}dx & v = \frac{1}{p}x^p \end{bmatrix}$$

Then, we have that

$$\Gamma(p) = \lim_{C\to 0^+}\int_C^\infty x^{p-1}e^{-x}dx =$$

$$\lim_{C\to 0^+}\left(e^{-x}\frac{1}{p}x^p\Big|_C^\infty - \frac{1}{p}\int_C^\infty -x^pe^{-x}dx\right)$$

Note that,

$$\lim_{C\to 0^+}\int_C^\infty x^pe^{-x}dx = \Gamma(p+1)$$

Then we see that

$$\Gamma(p) = \lim_{C\to 0^+}\left(e^{-x}\frac{1}{p}x^p\Big|_0^\infty\right) + \frac{1}{p}\Gamma(p+1)$$

Evaluating the first term, as the upper limit approaches ∞, we find that

$$\lim_{C\to 0^+}\left(e^{-x}\frac{1}{p}x^p\Big|_C^\infty \to 0\right)$$

Thus,

$$\Gamma(p) = \frac{1}{p}\Gamma(p+1)$$

Case 2: Consider when $p \in (0, 1)$.

Now, we pick u and dv such that,

$$\begin{bmatrix} u = e^{-x} & dv = x^{p-1}dx \\ du = -e^{-x}dx & v = p\frac{1}{p}x^p \end{bmatrix}$$

Then, for some $p - 1 \in \mathbb{Z}^{-1}$ we have that

$$\Gamma(p) = \lim_{C \to 0^+} \int_C^\infty x^{p-1}e^{-x}dx =$$

$$\lim_{C \to 0^+} \left(e^{-x}x^p \Big|_C^\infty - \frac{1}{p} \int_C^\infty -x^p e^{-x}dx \right)$$

By a similar reasoning as in Case 1, we find that the first term converges to 0.

Again we see that,

$$\Gamma(p) = \frac{1}{p}\Gamma(p+1)$$

We may rearrange our results from Case 1 and 2 to obtain the following,

$$\Gamma(p+1) = p\Gamma(p)$$

Which holds for $p < 0$. □

Next, recall that the factorial function $n!$, evaluated at some nonnegative integer n, is the found by taking the product of all nonnegative integers less than n. This idea leads us into our next lemma, which shows us the Gamma Function's relationship to the factorial function.

Lemma F:

For all $n \geq 0$, $\Gamma(n+1) = n!$.

Proof. Notice that for $p = 2$ the Gamma Function is evaluated to obtain,

$$\Gamma(2) = \lim_{C\to\infty} \int_0^C x^{2-1} e^{-x} dx$$

$$= \lim_{C\to\infty} \int_0^C x e^{-x} dx$$

$$= \lim_{C\to\infty} \left(-xe^{-x}\Big|_0^C + \int_0^C e^{-x} dx \right)$$

$$= -(0-1) - 0 = 1$$

Thus,

$$\Gamma(2) = 1$$

Observe that, by the recursive relation,

$$\Gamma(3) = 2\Gamma(2) = 2 = 2!$$

$$\Gamma(4) = 3\Gamma(3) = 3 \cdot 2 = 3!$$

$$\Gamma(5) = 4\Gamma(4) = 4 \cdot (3 \cdot 2) = 4!$$

Following this, we wish to show that this recursive relation holds for all natural n.

Suppose[1] that $\Gamma(n+1) = n!$ Next, consider $\Gamma(n+2)$.

We wish to show that $\Gamma(n+2) = (n+1)!$

By the recursive relation,

$$\Gamma(n+2) = (n+1)\Gamma(n+1)$$

[1]This is our induction hypothesis.

By the inductive hypothesis,

$$\Gamma(n+2) = (n+1)(n)! = (n+1)!$$

Thus, for any natural number n,

$$\Gamma(n+1) = n!$$

□

Aside from the factorial values of the Gamma function, there are other *interesting* values obtained from using the recursive relation.

Fractional Values of $\Gamma(p)$

In particular, the $\Gamma(p)$ may be evaluated for $p = \frac{1}{2}$, although this requires the use of a Jacobian transformation, which will be discussed when we introduce the Beta function. Evaluating the corresponding integrals, we are able to list several interesting values of the Gamma function evaluated at particular fractional values:

$$\begin{aligned}
\Gamma\left(\frac{1}{2}\right) &= \sqrt{\pi} \\
\Gamma\left(\frac{3}{2}\right) &= \frac{\sqrt{\pi}}{2} \\
\Gamma\left(\frac{5}{2}\right) &= \frac{3\sqrt{\pi}}{4} \\
&\vdots \\
\Gamma\left(n+\frac{1}{2}\right) &= \frac{(2n-1)!}{2^n}\sqrt{\pi}
\end{aligned}$$

Notice that we've only chosen to evaluate values of the Gamma function that are of the form $n + \frac{1}{2}$ for $n \in \mathbb{N}$. This was so because we can make use of the recursive relationship of the Gamma function in such a way that allows us to generate each successive value.

Moving further, we again consider the Gamma function,

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx$$

and we will observe that the integral is *not defined for the case where* $p \in \mathbb{Z}^-$.

We begin by first explaining why $p = 0$ is not defined, and see how the preceding statement follows from this.

Theorem 2.1: The Gamma Function is Not Defined For $p = 0$

Proof. We Begin by considering the Gamma function $\Gamma(p)$ for $p = 0$,

$$\begin{aligned}
\Gamma(0) &= \int_0^\infty x^{0-1} e^{-x} dx \\
&= \int_0^\infty x^{-1} e^{-x} dx \\
&= \int_0^1 x^{-1} e^{-x} dx + \int_1^\infty x^{-1} e^{-x} dx \\
&= \lim_{C_1 \to 0^+} \int_{C_1}^1 x^{-1} e^{-x} dx + \lim_{C_2 \to \infty} \int_1^{C_2} x^{-1} e^{-x} dx
\end{aligned}$$

Now, we pick u and dv such that,

$$\begin{bmatrix} u = e^{-x} & dv = x^{-1} dx \\ du = -e^{-x} dx & v = \ln x \end{bmatrix}$$

Integrating by parts we see that,

$$\begin{aligned}\Gamma(0) &= \lim_{C_1\to 0^+}\int_{C_1}^{1} x^{-1}e^{-x}dx + \lim_{C_2\to 0^+}\int_{1}^{C_2} x^{-1}e^{-x}dx \\ &= \lim_{C_1\to 0^+}\left(e^{-x}\ln x\Big|_{C_1}^{1} + \int_{C_1}^{1}\ln xe^{-x}dx\right) + \lim_{C_2\to\infty}\left(e^{-x}\ln x\Big|_{1}^{C_2} + \int_{1}^{C_2}\ln xe^{-x}dx\right)\end{aligned}$$

In particular, if we evaluate the limit of the first term,

$$\begin{aligned}\lim_{C_1\to 0^+} e^{-x}\ln x\Big|_{C_1}^{1} &= \lim_{C_1\to 0^+}\left(e^{-1}\ln(1) - e^{-C_1}\ln(C_1)\right) \\ &= \lim_{C_1\to 0^+} -e^{-C_1}\ln(C_1) \\ &= \lim_{C_1\to 0^+}\frac{-\ln(C_1)}{e^{C_1}} \Rightarrow \frac{-(-\infty)}{1} = \infty\end{aligned}$$

Hence, this limit does not exist.

Therefore, $\Gamma(p)$ is not defined for $p = 0$. □

We want to extend the Gamma function for $p < 0$ and we also want the recursive relation to still hold. Assuming this we find:

Theorem 2.2: The Gamma Function is Not Defined For $p \in \mathbb{Z}^-$

Proof. Consider that we may rewrite the recursive relation in the following way,

$$\Gamma(p) = \frac{\Gamma(p+1)}{p}$$

For the sake of contradiction, suppose that $\Gamma(-1)$ exists.

We see that, for $\Gamma(-1)$

$$\Gamma(-1) = \frac{\Gamma(-1+1)}{-1}$$

$$\Gamma(-1) = \frac{\Gamma(0)}{-1}$$

Recall that we have shown that the Gamma function is not defined for $p = 0$, that is, $\Gamma(0)$ is not defined.

Thus, $\Gamma(-1)$ is similarly not defined.

Inductively, we may show that the Gamma function is not defined for all $p \in \mathbb{Z}^-$.

Suppose[2] that $\Gamma(p)$ is not defined for $p \in \mathbb{Z}^{-1}$. Then consider $\Gamma(p+1)$.

By the Recursive relation, we see that $\Gamma(p+1) = \Gamma(p) \cdot p$.

By the inductive hypothesis, $\Gamma(p)$ is not defined. Thus, $\Gamma(p+1)$ is not defined.

Thus, $\Gamma(p)$ is not defined for all $p \in \mathbb{Z}^{-1}$. $\square$

So far we've defined the Gamma function to be well-defined for nonnegative values, but not defined for negative integers. We point our attention to the following graph of $y = \Gamma(x)$:

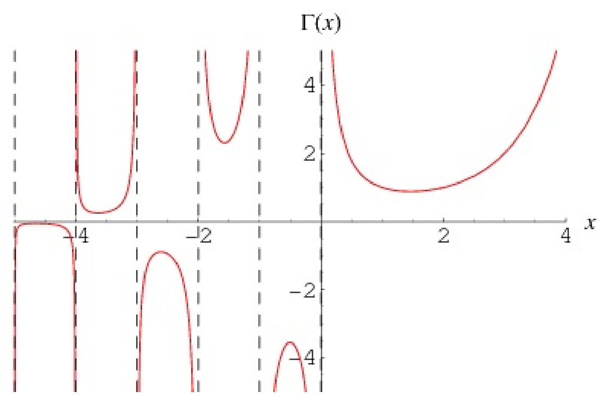


Figure 2.2

The asymptotic behavior of the graph reassures us of the fact that the function is not defined for integers less than 1, but also raises more questions. The values of $\Gamma(x)$ seem to alternate between positive and negative values for $-n-1 < x < -n$ where $n \in \mathbb{N}$.

Performing calculations similar to evaluating $\Gamma(\frac{1}{2})$ reflects the left hand behavior of this graph.

[2]This is our induction hypothesis.

The Negative fractional Values of Γ

By evaluating for $\Gamma(\frac{-1}{2})$, and again using the recursive relation $\Gamma(p) = \frac{1}{p}\Gamma(p+1)$, we arrive at some interesting values of Γ for $p < 0$:

$$\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$$
$$\Gamma\left(-\frac{3}{2}\right) = \frac{4\sqrt{\pi}}{3}$$
$$\Gamma\left(-\frac{5}{2}\right) = -\frac{8\sqrt{\pi}}{15}$$
$$\vdots$$
$$\Gamma\left(\frac{1}{2} - n\right) = \frac{(-1)^n \cdot 4^n n!}{(2n)!}\sqrt{\pi}$$

Here we see $\Gamma(p)$ evaluated for non-integer values, which are 1 apart form each other, alternating in sign. This pattern is clearly reflected in Figure 2.2. It should be noted that, just as before, evaluation of $\Gamma(\frac{-1}{2})$ requires the use of a Jacobian transformation, which is reserved for a later discussion.

Proof of Life... or At Least Evidence of "0!"

The notion of 0! representing the empty product was introduced briefly before, but this serves to be an interesting topic of discussion among high-school sophomores, who are just learning about the significance of factorial function. While there are a variety of ways that $0! = 1$ can be explained, this is an idea that is very well accepted.
A common tactic employed by teachers is to propose the question: *If I have zero items, how many ways may I arrange my items? Just one.*

Here, we offer just one more:

Lemma G: $\Gamma(1) = 1$ shows us that $0! = 1$.

Proof. Consider the Gamma function evaluated at $p = 1$, which is indeed greater than 0. According to the recursive formula,

$$\Gamma(1) = 0\Gamma(0)$$

Further,

$$\Gamma(2) = 1\Gamma(1)$$

So we have that,

$$\Gamma(1) = 0!$$

Then consider that

$$\begin{aligned}
\Gamma(1) &= \int_0^\infty x^{1-1}e^{-x}dx = \int_0^\infty e^{-x}dx \\
&= \lim_{c\to\infty}\int_0^c e^{-x}dx = \lim_{c\to\infty} -e^{-x} \\
&= \lim_{c\to\infty} -e^{-c} + e^0 = \lim_{c\to\infty} -\frac{1}{e^c} + 1 \\
&= 0 + 1 = 1
\end{aligned}$$

Thus, if the values of the Gamma function do indeed coincide with that of the Factorial function, then this implies that $0! = 1$. □

Chapter 3

Applications in Mathematics and the Physical Sciences

The work of Euler has most definitely served as inspiration for further developments in the fields of mathematics and physics, in particular. The remained of the paper will focus on applications of the Gamma Function in these fields. Notable applications of Γ include:

- Calculating the volumes of n-dimensional unit-balls
- Using the Gamma Distribution to determine the wait time for the αth occurrence of a Poisson distributed event.
- Simplifying calculations for the Beta Function
- Calculating the αth derivative of a monomial, for any $\alpha \in \mathbb{R}$.

3.1 The Unit-Ball

It might pleasing to discover that a simple understanding of concepts such as the area of a circle and volume of a sphere fits in perfectly, and may even be derived from a more generalized, slightly more complex concept. The concept which we are referring to is the *volume of an n-dimensional unit ball.*

The volume of a unit ball in n-dimensions, with radius 1, is given by the following formula[1],

$$V_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$$

3-Dimensional Unit Ball

When we think of volumes in the more intuitive sense, we are likely to limit our focus to the three dimensional case of a sphere, who's volume is given by the formula $\frac{4}{3}\pi r^3$. For a unit sphere of radius 1, we calculate the volume this way to be $\frac{4}{3}\pi(1^3) = \frac{4}{3}\pi$

With a bit of algebra, and with the aid of the Gamma Function, we are able to derive this from the general case above. Setting $n = 3$ we have,

$$\begin{aligned} V_3 &= \frac{\pi^{\frac{3}{2}}}{\Gamma(\frac{3}{2}+1)} \\ &= \frac{\pi^{\frac{3}{2}}}{\Gamma(\frac{5}{2})} \\ &= \frac{\pi^{\frac{3}{2}}}{\frac{3\sqrt{\pi}}{4}} \\ &= \frac{4}{3}\pi \end{aligned}$$

We see that the value we obtained corresponds to the volume we calculated previously.

2-Dimensional Unit Ball

It should be first noted that referring to the volume of an object in 2 dimensions simply refers to the area of the figure. Recall that the area of a circle in 2 dimensions is given by

[1] Joel, Azose. "On the Gamma Function and its Applications."

the formula πr^2. For the unit circle, with radius 1, the area will be $\pi(1^2) = \pi$.

Similar as before, setting $n = 2$ for the 2 dimensional case we have,

$$\begin{aligned} V_2 &= \frac{\pi^{\frac{2}{2}}}{\Gamma(\frac{2}{2}+1)} = \frac{\pi}{\Gamma(2)} \\ &= \frac{\pi}{1} = \pi \end{aligned}$$

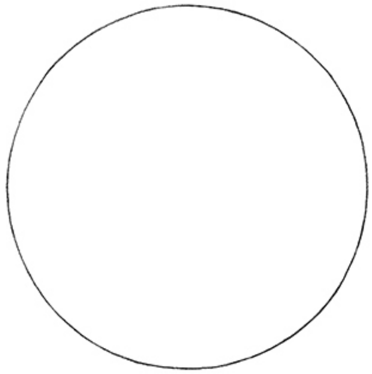
Recall that the circle with radius of 1 contains all of the points that are 1 unit away from the center, which also serving as a boundary for all points that are within 1 unit from the center. We can say the same is true for the unit sphere in 3 dimensions. With these examples in mind, consider that such shapes allow us to understand and quantify distances between points in a space.

Now that we've provided ourselves with a new way to remind ourselves of two important formulas from geometry, let us now see what results from extending these concepts beyond our intuition and consider the volume of a 4th dimensional unit ball.

4-Dimensional Unit Ball

Consider that for $n = 4$ we have,

$$\begin{aligned} V_4 &= \frac{\pi^{\frac{4}{2}}}{\Gamma(\frac{4}{2}+1)} \\ &= \frac{\pi^2}{\Gamma(3)} \\ &= \frac{\pi^2}{2} \\ &= \frac{1}{2}\pi^2 \end{aligned}$$



Circle in 2D

Sphere in 3D

Continuing the process which we have started, determining volumes of unit spheres in higher and higher dimensions, we are similarly able to extend form an understanding of spacial distances in these higher dimensions.

3.2 The Gamma Distribution

A very popular application of the Gamma Function arises in the study of statistical analysis. This application truly arises from the usefulness of the factorial function in describing discrete probability distributions, in which events are occurring discretely, not continuously, over an interval.

Much like the way in which the Gamma Function is a continuous extension of the Factorial Function, we find that the Gamma Distribution is an extension of such discrete probability distributions.

The Gamma Distribution belongs to a family of continuous probability distributions of *two parameters*, α & θ and is used in *probability and statistics.*

The random variable X will have a **gamma distribution** if it's probability density function for the αth occurrence is defined by

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{\frac{-x}{\theta}}$$

where θ is the mean wait time for the first occurrence of an event.

The usefulness of the Gamma Function in the Gamma Distribution is most clearly seen through an example. We would find the Gamma Distribution to be of much use to us in the following scenario:

Given that the person A wins a jackpot, gambling on a penny-slot machine, every 3 years, we might wish to use the Gamma Distribution if we were concerned with finding the probability that person A will wait 5 years before earning a jackpot. To do so, we simply determine the values of α and θ and begin to evaluate for the function above. From this, person A would be advised whether or not to continue gambling for the next 5 years.

From this fictional example, as humorous as it might seem, we are able to see just how the applications of the Gamma Function provides us with a means of understanding larger questions.

3.3 The Beta Function

Earlier, it was mentioned that the Gamma Function can be extended to the complex number system. The Beta Function is our first example of this, yet another discovery of Euler[2]!

The Beta Function itself is has several notable applications, giving us binomial coefficients, models for scattering amplitude (which is important for study of particle physics), and a model for the distribution of wealth (an application useful to socio-economics).

With this in mind, it is useful to see that such an important function may be expressed in terms of the Gamma Function. We start by defining the Beta Function in the following way:

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$$

which is defined for $x, y \in \mathbb{R}^+$ such that the real components of x and y are positive.

Note that this function is a function of two variables and are plotted in the three-dimensional plane in figures 3.3a (as a function of two variables) and 3.3b (as a contour plot). In both cases, the asymptotic behavior should be reminiscent of the plot of $y = \Gamma(x)$. We intend to explain the relationship between these functions.

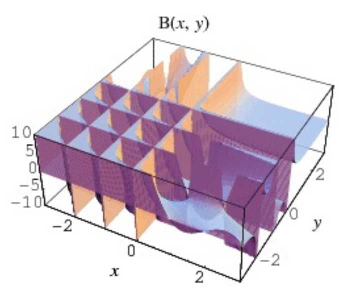


Figure 4.1a

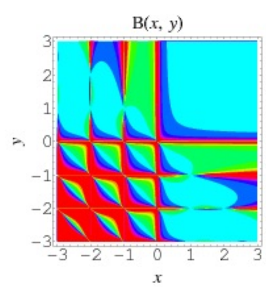


Figure 4.1b

(Obtained from *Wolfram MathWorld*)

[2]Artin, The Gamma Function

Theorem 3.3

The Beta function can be expressed in the following way[3],

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

An evaluation of the Beta function, as it was previously defined, will allow us to show that the theorem above holds. It should be noted that this proof requires knowledge of Jacobian transformations, so we shall introduce the aspects of this method that are of use to us.

We are able to use a Jacobian Matrix to perform the following transformation,

$$\int\int f(x,y)dxdy = \int\int \omega f(\omega\cos(\theta), \omega\sin\theta)d\omega dr$$

Where we determine ω by the evaluating the determinate of the following matrix,

$$\begin{bmatrix} \frac{\partial x}{\partial \omega} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \omega} & \frac{\partial y}{\partial \theta} \end{bmatrix}$$

With this, we are able to prove Theorem 3.3

Proof. Consider the product of $\Gamma(x)$ and $\Gamma(y)$, for $x, y > 0$.

[3]Riddhi, D. "Beta Function and its Applications"

We see that,

$$\begin{aligned}\Gamma(x)\Gamma(y) &= \int_0^\infty u^{x-1}e^{-u}du \int_0^\infty v^{y-1}e^{-v}dv \\ &= \int_0^\infty \int_0^\infty v^{y-1}u^{x-1}e^{-u}e^{-v}dudv \\ &= \int_0^\infty \int_0^\infty v^{y-1}u^{x-1}e^{-u-v}dudv\end{aligned}$$

We continue by substituting, letting $u = \omega cos^2(\theta)$, $v = \omega sin^2(\theta)$, and utilizing the Jacobian transformation of these functions,

$$\begin{aligned}\Gamma(x)\Gamma(y) &= \int_0^\infty \int_0^\infty v^{y-1}u^{x-1}e^{-u-v}dudv \\ &= 2\int_{\omega=0}^\infty \int_{\theta=0}^{\frac{\pi}{2}} e^{-\omega}\omega^{x+y-1}\cos^{2x-1}(\theta)sin^{2y-1}(\theta)d\omega d\theta \\ &= 2\int_{\omega=0}^\infty e^{-\omega}\omega^{x+y-1}d\omega \int_{\theta=0}^{\frac{\pi}{2}} \cos^{2x-1}(\theta)sin^{2y-1}(\theta)d\theta \\ &= 2\Gamma(x+y)\int_{\theta=0}^{\frac{\pi}{2}} \cos^{2x-1}(\theta)sin^{2y-1}(\theta)d\theta\end{aligned}$$

Substituting again, setting $t = \cos^2(\theta)$ and $dt = 2\sin(\theta)\cos\theta d\theta$,

$$\begin{aligned}\Gamma(x)\Gamma(y) &= 2\Gamma(x+y)\int_{\theta=0}^{\frac{\pi}{2}} \cos^{2x-1}(\theta)sin^{2y-1}(\theta)d\theta \\ &= \Gamma(x+y)\int_{t=0}^{1} t^{x-1}(1-t)^{y-1}dt \\ \Gamma(x)\Gamma(y) &= \Gamma(x+y)\beta(x,y)\end{aligned}$$

Dividing, we see indeed that

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

□

Speaking to the inter-relatedness and usefulness of the Gamma and Beta functions, Mary Boas puts it this way, "Whenever you want to evaluate a B function, you use [Theorem 3.3] first and then look up values for the Γ function in the tables."[4]. From this application, we see the Gamma Function again playing the role of a tool for simplification.

3.4 Fractional Calculus

We have now arrived at our last topic of discussion, a field of mathematics that extends the notion of differentiation beyond integer-indexed derivatives, *Fractional Calculus.*

The **"fractional" derivative** of a function with respect to x is denoted by $\frac{dt^\alpha}{dx^\alpha} = D^n_\alpha$ for some real $\alpha \in \mathbb{R}$ and satisfies the following properties[5]:

$$D^\alpha_x[Cf(x)] = CD^\alpha_x[f(x)] \text{ for some constant } C$$
$$D^\alpha_x[f(x) \pm g(x)] = D^\alpha_x[f(x)] \pm D^\alpha_x[g(x)]$$
$$D^\alpha_x[f(x) \cdot g(x)] = \Sigma^n_{k=0} \binom{\alpha}{k} D^{\alpha-k}_x[g(x)] D^\alpha_x[f(x)]$$

We should note that these properties to hold for the case where $\alpha \in \mathbb{N}$, and are often taught in introductory calculus courses.

Recent developments in mathematics and physics rely on this idea of fractional derivatives, despite its recent emergence (relative to the other applications mentioned thus far).

Fractional calculus has applications in the studies of fluid flow, diffusive transport, electrical networks, electromagnetic theory, and even probability.

We should mentioned that specific details of this field are well beyond the scope of this paper, but we do limit our focus enough to begin to grasp this concept. We begin here, with

[4]Boas, Mathematical Methods in the Physical Sciences
[5]Bologna, Mauro "Fractional Calculus"

using the following theorem, in which we are dealing with a monomial.

Theorem 4.3

For $\alpha \in \mathbb{R}$ or $\alpha \in \mathbb{C}$, the real-indexed derivative of a monomial x^β is[6]

$$\frac{d^\alpha}{dx^\alpha}[x^\beta] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^\beta - \alpha$$

Although the proof of this theorem is not given here, it relies entirely on the ease of working with monomials and the presence of the binomial coefficient $\binom{n}{k}$ that appears in the last of the properties listed above. We should also note that, this theorem may easily be extended to constant multiples, sums, and product of monomial expressions, by the properties mentioned previously.

We may begin by examining when $\alpha = 1$ and $\beta = 2$, which is the the 1st derivative of the monomial x^2 with respect to x,

$$\begin{aligned} D_x^1[x^2] &= \frac{d^\alpha}{dx^\alpha}[x^2] \\ &= \frac{\Gamma(2+1)}{\Gamma(2+1-1}x^{2-1} \\ &= \frac{2!}{1!}x \\ &= 2x \end{aligned}$$

Through a more basic use of calculus, we would similarly determine that $\frac{d}{dx}(x^2) = 2x$.

Interestingly, this notion of real-indexed derivatives brought us to an understanding of what a fractional derivative might provide for us, but now we turn our attention towards negative-indexed derivatives.

Consider when $\alpha = -1$ and $\beta = 2$, which is the the 1st derivative of the monomial x^2

[6] Bologna, 2014

with respect to x,

$$\begin{aligned}
D_x^{-1}[x^2] &= \frac{d^\alpha}{dx^\alpha}[x^2] \\
&= \frac{\Gamma(2+1)}{\Gamma(2+1-(-1))}x^{2-(-1)} \\
&= \frac{\Gamma(3)}{\Gamma(4)}x^{2-(-1)} \\
&= \frac{2!}{3!}x^3 \\
&= \frac{2}{3\cdot 2}x^3 \\
&= \frac{1}{3}x^3
\end{aligned}$$

Now recall that,

$$\int x^2 dx = \frac{1}{3}x^3 + C$$

for some real constant C.

It should be noted that there are certain limitations for the domain of the Gamma Function that restrict our ability to evaluate the negative derivative of such a function, since whenever $(\beta - \alpha) \in \mathbb{Z}^-$ we have that $\Gamma(\beta + 1 - \alpha)$ is not defined, since it is at most $\Gamma(0)$ in this case.

It is for this reason that alternative expressions of fractional derivatives arise for particular functions, but even in cases other than monomials, the Gamma Function continues serve not only as a means of making calculations possible, but it also allows us to bridge the gap between our previous understanding and such a complex concept.

References

1. Bologna, Mauro. "Fractional Calculus." An Introduction for Physicists Fractional Calculus (2014): 43-68. Web. 1 Apr. 2016.

2. Boas, Mary L. Mathematical Methods in the Physical Sciences. New York: Wiley, 1966. Print.

3. Berg, Christian. "Integral Representation of Some Functions Related to the Gamma Function." Mediterranean Journal of Mathematics MedJM 1.4 (2004): 433-39. Web. 1 Mar. 2016.

4. Buck, R. Creighton, and Ellen F. Buck. Advanced Calculus. New York: McGraw-Hill, 1965. Print.

5. Artin, Emil. The Gamma Function. New York: Holt, Rinehart and Winston, 1964. Print.

6. Joel, Azose. "On the Gamma Function and its Applications." University of Wshington. Web. Mar. 1 2016.

7. Riddhi, D. "Beta Function and its Applications." The University of Tennessee. Web. Mar. 13 2016.

8. Viochita, Adriana "On the History of the Gamma Function." Review of Economic Studies & Research Virgil Madgearu. Web. Mar. 1 2016.