Anh Maciag

Saint Mary's College of California

Department of Mathematics

May 16, 2016

A Two-Person Bargaining Problem

Supervisors:
Professor Kathryn Porter
Professor Michael Nathanson
Professor Chris Jones

Contents

1 Abstract

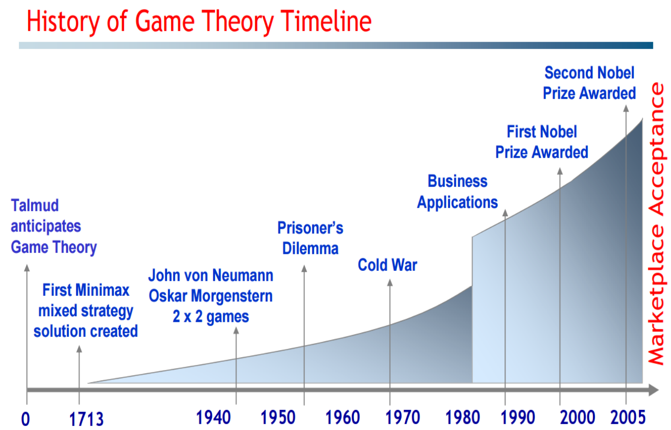
A two-person bargaining problem is a problem involving two individuals/players who possess the opportunity to collaborate for mutual benefit in more than one way. It is a situation of understanding how two players should cooperate when non-cooperation leads to Pareto-inefficient results. This means that cooperation can bring better results that players would be more satisfied than if they were not cooperating.

2 Story

Imagine there are two friends Tom and Chris who possess different tastes in evening entertainment. Tom has been watching Economics News alone for awhile and has realized he does not like watching TV by himself. His friend, Chris, has been enjoying watching Shark Tank on his own. Even though Chris likes doing it alone, he would still prefer to watch TV with somebody else. One day, Tom and Chris meet and decide to hang out and watch their TV show's preference together. So how should they cooperate so that both of them would be happy with the TV show's choice? It is a situation of conflict of interests between two friends, which is an example of Game Theory.

3 Game Theory

Game Theory is a branch of mathematics that analyzes games, especially situations including two parties with conflicting interests.



Source: Open Options Introduction To Game Theory

The origin of Game Theory started in the Talmud's time a millennium ago. There was a man who owes a lot of debt, but dies with not enough money to pay everyone. So the problem was how should his inheritance be divided so that the collectors were satisfied with the portion of money received? That was one of the earliest discussions of analyzing conflict of interests among parties and their behaviours.

However, the first formal study of Game Theory was conducted and published in 1944 by John von Neumann and Oskar Morgenstern in their work called **Theory of Games and Economic Behaviour**. There are many game types, but the focus of this paper is the Two-Person Bargaining Problem.

4 A Two-Person Bargaining Problem

As mentioned before, a two-person bargaining problem is a problem of understanding how to players should cooperate so that both of them would be satisfied with the results from the cooperation. It involves cooperation which is a situation wherein the two players sit down on and make decisions together. If they make decisions independently, it would be a non-cooperation. The payoff from a cooperation for each player is greater than the payoff for each individual from non-cooperation. Now, let's try to find a mathematical way to solve the two-person bargaining problem.

5 Set-up

5.1 Happiness Score

Now, let's try find a mathematical way to solve the bargaining problem between Tom and Chris. We want to find out how often they should watch together Economics News and Shark to maximize their emotional satisfaction. In order to do it, we need to make certain assumptions to model mathematically the situation between the two friends. Assume that a person's payoff is represented as a happiness score which is an emotional satisfaction from watching a TV show's preference. Considering the example of Tom and Chris, let the point (u, v) be a pair of payoffs where u is a happiness score for Tom and v is a happiness score for Chris. Suppose that happiness score can be quantified among a few values: $5, 1, 1/2, 0$ where 0 is the lowest happiness score and 5 is the highest one. The individual happiness score depends on certain assumed conditions as follows.

- Watching own choice of the entertainment together is 5
- Watching the other person's choice together is 1
- Watching the friend's choice alone is 0
- Tom watching his own choice alone is 0
- Chris watching his own choice alone is 1/2

Note that the above bullet points are assumed stated conditions and the emotional satisfaction is quantified at discretion. If we interviewed Tom and Chris personally, it would be hard to decide their happiness score. However, for simplicity, let's suppose the above happiness score values reflect Tom's and Chris' emotional satisfaction assigned to their decision.

5.2 Pure Joint Strategy

Suppose that the decision how both players should play is based on their happiness scores. The way that one player decides to play is called a **a pure**

strategy. It defines a particular action that an individual player follows. Considering the example of two friends, if Tom decides to watch Economics News, then watching Economics News would be a pure strategy. So, let E be a strategy to watch Economics News and S be a strategy to watch Shark Tank. Having two pure strategies, we can create a **a pure joint strategy**, which is a pair of pure strategies of the two players. Let E and S be pure strategies either for Tom or Chris. Then, the possible joint strategies are as follows

$$(E, S), (S, E), (E, E), (S, S)$$

Let Tom and Chris represent the first and second components in the pure joint strategy respectively.Then (E, S) shows that Tom decides to watch Economics News whereas Chris decides to watch Shark Tank. On the other hand, (E, E) indicates that both of them decide to watch Economics News.

5.3 Payoff Matrix

The situation of conflicting interests between Tom and Chris can be put in the **payoff matrix**. The payoff matrix is a bimatrix showing all the possible outcomes between the two players and the pair of payoffs for them would be the entries of that matrix. Therefore, a **payoff matrix** is a visual representation of all the possible results that can occur when two people have to make a strategic decision. That decision is considered as a strategic decision because each person making decision has to take into consideration how his or her choice will influence his or her opponent's choice and how his or her opponent's choice will influence his or her own choice. The below payoff matrix represents the example of two friends

$$\begin{array}{cc} & \text{Chris} \\ & \begin{array}{cc} E & S \end{array} \\ \text{Tom} & \begin{array}{c} E \\ S \end{array} \begin{pmatrix} (5,1) & (0,1/2) \\ (0,0) & (1,5) \end{pmatrix} \end{array}$$

where Tom is the row player whereas Chris is the column player. The entry $(5, 1)$ in the matrix shows both Tom and Chris agree to watch Economics News together. Based on the previous stated conditions, Tom's happiness score is 5 whereas Chris' is 1. On the other hand, the entry $(0, 1/2)$ would indicate Tom and Chris want to watch their own entertainment. This can lead to disagreement between the friends and they may end up watching their show on their own. As the result, Tom would generate his happiness score of 0 and Chris's happiness score would 1/2.

Splitting up the payoff matrix, the individual payoff matrix would be created as follows

$$\mathrm{T} = \begin{array}{c} \\ E \\ S \end{array} \begin{array}{c} \begin{array}{cc} E & S \end{array} \\ \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \qquad \mathrm{C} = \begin{array}{c} \\ E \\ S \end{array} \begin{array}{c} \begin{array}{cc} E & S \end{array} \\ \begin{pmatrix} 1 & 1/2 \\ 0 & 5 \end{pmatrix} \end{array}$$

where T is an individual payoff matrix for Tom and C is an individual payoff matrix for Chris. Even though two matrices have been produced, the payoff assigned to a particular strategy for each player remains the same.

5.4 Mixed Joint Strategy

In a cooperative environment when two players would sit down and try to make a decision together, they would collaboratively assign a probability to each pure mixed strategy that they would play with. Thus, a **mixed joint strategy** would arise and the definition is as follows.

Let $\vec{\pi}$ be a two-person game with $m \times n$ payoff matrices A and B. A **joint strategy** is an $m \times n$ probability matrix $P = (p_{ij})$ such that

$$p_{ij} \geq 0, \text{ for } 1 \leq i \leq m,\ 1 \leq j \leq n, \text{ and } \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} = 1.$$

The mixed joint strategy is just a probability matrix P such that each entry is a probability assigned to a pair off payoffs of a particular pure mixed strategy. Thus, the sum of all the entries has to add up to 1. Let's look at some examples of the mixed joint strategy below.

$$\mathrm{P} = \begin{array}{c} \\ E \\ S \end{array} \begin{array}{c} \begin{array}{cc} E & S \end{array} \\ \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \end{array} \qquad \mathrm{P}' = \begin{array}{c} \\ E \\ S \end{array} \begin{array}{c} \begin{array}{cc} E & S \end{array} \\ \begin{pmatrix} 1/2 & 1/8 \\ 1/8 & 1/4 \end{pmatrix} \end{array}$$

Assume that Tom and Chris agree to play accordingly with the mixed joint strategy P. Notice that the entries on the diagonal are $1/2$. This means that both friends agree to watch half of the time Economics News and Shark Tank. Now, suppose they agree to play accordingly with the mixed joint strategy P' and look again at the diagonal entries. They agree to watch more often Economics News than Shark Tank. Since watching Economics News would increase Tom's happiness score, the matrix P' is more advantageous to Tom than Chris. Regardless of the choice of the mixed joint strategy, the summation of all entries will always be 1.

5.5 Expected Payoff

Given a payoff matrix and a mixed joint strategy, we can evaluate the **expected payoff** for a particular player.

Let P be a mixed joint strategy. The **expected payoff** π for a particular player is defined as

$$\pi(P) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} a_{ij}$$

where a_{ij} is the payoff of that pure strategy and p_{ij} is the probability assigned to that strategy. Considering the example of Tom and Chris, suppose that they agree play accordingly to the mixed joint strategy P' and the two individual payoff matrices for the two players are known as follows

$$\mathrm{P}' = \begin{array}{c} \\ E \\ S \end{array} \begin{array}{c} \begin{matrix} E & S \end{matrix} \\ \begin{pmatrix} 1/2 & 1/8 \\ 1/8 & 1/4 \end{pmatrix} \end{array} \qquad \mathrm{T} = \begin{array}{c} \\ E \\ S \end{array} \begin{array}{c} \begin{matrix} E & S \end{matrix} \\ \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \qquad \mathrm{C} = \begin{array}{c} \\ E \\ S \end{array} \begin{array}{c} \begin{matrix} E & S \end{matrix} \\ \begin{pmatrix} 1 & 1/2 \\ 0 & 5 \end{pmatrix} \end{array}$$

Let $\pi_T(P)$ be the expected payoff for Tom and $\pi_C(P)$ be the expected payoff for Chris. Following the above defintion of expected payoff

$$\pi_T(P') = (\tfrac{1}{2})5 + (\tfrac{1}{8})0 + (\tfrac{1}{8})0 + (\tfrac{1}{4})1 = 2.75$$

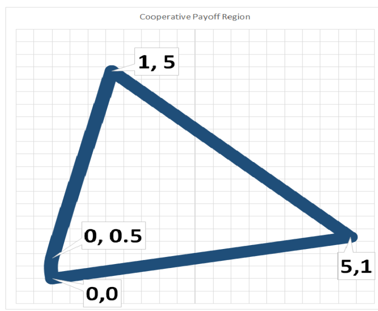
$$\pi_C(P') = (\tfrac{1}{2})1 + (\tfrac{1}{8})(1/2) + (\tfrac{1}{8})0 + (\tfrac{1}{4})5 \approx 1.81$$

Under this joint strategy P', the expected payoff for Tom is 2.75 and for Chris is 1.81. It makes sense that Tom's expected payoff is greater than Chris' because the mixed joint strategy P' is advantageous to Tom.

5.6 Cooperative Payoff Region

Choosing various joint strategies, we can get many different pair of expected for Tom and Chris. This can create the **cooperative payoff region** in the two dimensional plane consisting of the pairs of expected payoffs as points.

Formally, the **cooperative payoff region** is the set $\{\pi_1(P), \pi_2(P) : \mathbf{P}$ is a joint strategy$\}$ in R^2. In addition, the cooperative payoff region is closed, convex and bounded sets as the properties. Let's consider the cooperative region for Tom and Chris.



Using the pair of payoffs from the payoff matrix, the cooperative payoff region for Tom and Chris is created as above. We see that it is closed because every point in the region has upper and lower bounds. The region is bounded because it has boundaries and is being enclosed in a circle in R^2. In addition, it is convex because any chosen two points in the region, the line segment from these points would still lie in that region.

5.7 Criteria and Pareto Optimality

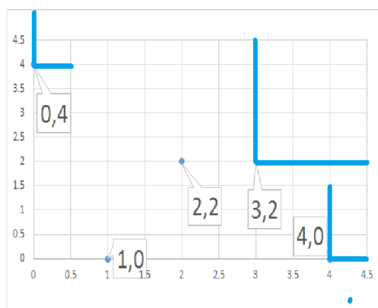
When the players cooperate with each other, they need to agree on which joint strategy to adopt. Therefore, they should pay much attention to two important criteria.

1. The payoff pair from the joint strategy is greater than or equal to payoff pair from non-joint strategy. Otherwise, the players would not want to collaborate as collaboration brings worse results.

2. The payoff pair produced from the joint strategy and agreed by them should be Pareto optimal

The **Pareto Optimal** pair is a pair of payoffs for two players such that it is impossible for one of the players to improve his payoff without making the other's payoff worse.

Let (u^*, v^*) be a pair of payoffs in K. Then (u^*, v^*) is **Pareto Optimal** if $u^* \geq u$ or $v^* \geq v$ for any (u, v) in K. For example, let $(3, 2)$, $(2, 2)$, $(1, 0)$, $(4, 0)$, $(0, 4)$ be in K. Plotting these points in the graph, we would have as below.

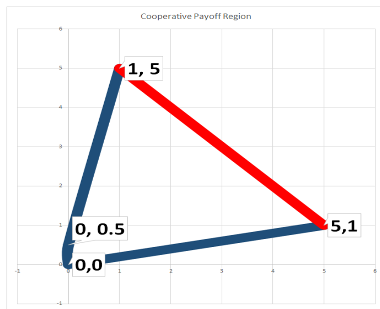


The perpendicular lines in the graph show the pairs of payoffs should go far up and far right in order to be Pareto Optimal. It can be seen that $(1, 0)$ is less than $(4, 0)$ since $(4, 0)$ is further to the right than $(1, 0)$. Using similar approach, the pair $(3, 2)$ is greater than $(2, 2)$. However, the points $(0, 4), (4, 0), (3, 2)$ are not necessarily greater than one another. The point $(0, 4)$ is further up than point $(4, 0)$. However, $(4, 0)$ is further to the right than $(0, 4)$. On the other hand, $(3, 2)$ is further up to the right than $(4, 0)$. But, $(4, 0)$ is further up than $(3, 2)$. Similarly with $(0, 4)$. Since, none of these pairs of payoffs are greater than one another, $(0, 4), (4, 0), (3, 2)$ are Pareto Optimal.

5.8 Bargaining Set

A collection of all Pareto Optimal pairs is the **bargaining set** which can be defined as follows.

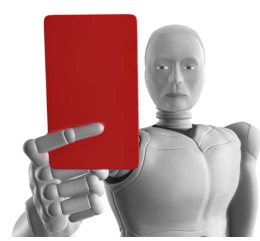
The bargaining set for a two-person cooperative game is the collection of all Pareto optimal payoff pairs (u, v) such that $u > u_1$ or $v > v_1$ where (u_1, v_1) is any other pair of payoffs. Let's consider the bargaining set of Tom and Chris.



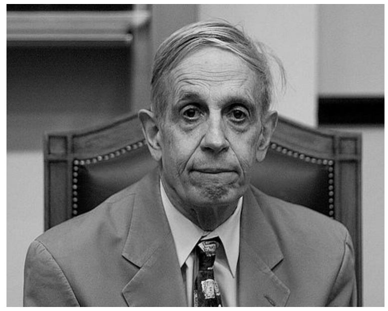
The red line in the graph is the bargaining set. Every point on that line segment is far up and far to the right. So, each point from that red line is greater than any point far left and far down. So, the problem is now which one Pareto Optimal Pair from the bargaining set Tom and Chris need to decide on.

6 Nash Bargaining Axioms

We need an arbiter who would attempt to make people involved in a conflict to come to an agreement.



It does not matter whether the arbiter is a human or a robot if the arbiter has a certain kind of qualities or a way judging that would be consistent. These qualities are represented by the Nash Bargaining Axioms introduced by John Nash.



John Forbes Nash (June 13, 1928- May 23, 2015) was an American mathematician and economist who made huge and fundamental contributions differential geometry, the study of partial differential equations and game theory. He received two major awards: Nobel Memorial Prize in Economic Sciences and Abel Prize. There is even his biography as a film called *A Beautiful Mind*.

The theory developed by Nash gives a fair method of deciding which payoff pair in the bargaining set should be chosen. The idea is to show the existence of an arbitration procedure that, when given a payoff region P and a status quo point $(u_0, v_0) \in P$, will create a payoff pair called the arbitration pair which would be fair to both players.

Let an arbiter be an **arbitration procedure** Ψ such that with payoff region K and status quo point (u_0, v_0) will produce an optimal pair of payoffs (u^*, v^*) for both players.

$$\Psi(K, (u_0, v_0)) = (u^*, v^*)$$

where Status Quo point (u_0, v_0) is the initial pair of payoffs which players will receive if they cannot agree. In the case of Tom and Chris, these payoffs are the ones before they decide to watch their TV shows together.

As mentioned before, the arbiter needs a way of judging and/or a certain list of qualities that would make a consistent and fair decision to choose one Pareto Optimal pair from the bargaining set. These qualities are the Nash Bargaining Axioms as follows.

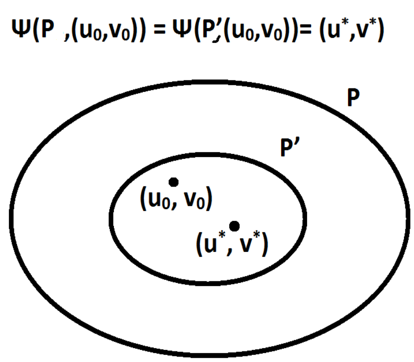
1. Individual Rationality: Let (u^*, v^*) be an arbitration pair and (u_0, v_0) be the status quo point. $u^* \geq u_0$ and $v^* \geq v_0$. Axiom 1 says that the

arbitration pair has to be greater than the **status quo point**. The Status quo point is the greatest pair of payoffs that the players can be sure to get without knowing the actions of each other. We want an arbiter to choose that arbitration pair. Otherwise, the players would not want to cooperate if the payoffs from the cooperation were lower than the ones resulted from non-coperation.

2. Pareto Optimality: Let (u^*, v^*) be an arbitration pair. (u^*, v^*) is Pareto optimal. We want an arbitration pair to be a pair of payoffs for two players such that it is impossible for one of the players to improve his payoff without making the other's payoff worse. Generally, Axiom 2 highlights that the arbitration pair is in the bargaining set.

3. Feasibility: $(u^*, v^*) \in P$. The arbitration pair should be in the cooperative payoff region P in order to be considered.

4. Independence of Irrelevant Alternatives: If P' is a payoff region contained in P and both (u_0, v_0) and (u^*, v^*) are in P',then $\Psi(P', (u_0, v_0)) = (u^*, v^*)$. Axiom 4 mentions that the solution of the bargaining problem would not change as the feasible region or set of outcomes is reduced as long as there are the same status quo point and the arbitration pair (See the image below).



5. Invariance Under Linear Transformations: Suppose P' is obtained from P by the linear transformation $u' = au + b$, $v' = cv + d$ where $a, c > 0$. Then $\Psi(P', (au_0 + b, cv_0 + d)) = (au^* + b, cv^* + d)$. Axiom 5 says

that there is no important change in the arbitration pair regardless of a change in units of the payoffs. In other words, it does not matter how the payoffs were measured whether in the happiness score, dollar amount or in the number of apples. The essential arbitration pair remains the same and the linear transformation just shifts how essential arbitration pair should be measured.

6. Symmetry: Suppose that P is symmetric (that is $(u, v) \in P$ if and only if $(v, u) \in P$), and that $u_0 = v_0$. Then $u^* = v^*$. Axiom 6 highlights that it does not matter how the players are labeled. If switching the labels between the two players leaves the bargaining problem remains the same, then the solution should be symmetrical. So if the players possess roles that are symmetric regarding the payoff region and the status quo point, they should receive the same payoff.

7 Nash's Thereoem

Having defined the Nash Bargaining Axioms as the qualities of the arbiter or the arbitration procedure, let's introduce the following very significant Nash's Theorem.

Theorem. There is a unique arbitration procedure Ψ satisfying Nash's axioms.

It is important that the arbitration procedure is unique. We want an arbiter to be unique such that the same one arbiter resolves the conflicting interests between two players. If we have two arbiters, they would have a different way of judging and produce two different arbitration pairs.

Before proving Nash Theorem, let's introduce a lemma regarding uniqueness that would help showing the proof of the Nash Theorem later.

Lemma (Uniqueness). Let K be a payoff region and $(u_0, v_0) \in K$. Suppose that there exists a point $(u, v) \in K$ such that $u > u_0$, $v > v_0$ and let H be the set of all $(u, v) \in K$ satisfying these inequalities. Define, on H, $f(u, v) = (u - u_0)(v - v_0)$. Then f attains its maximum on H at one and only point.

Proof. (Morris, 142) Let $H = \{(u, v) \in K : u \geq u_0, v \geq v_0\}$ be a closed,convex and bounded set . By a theorem of mathematical analysis, the function f, being continuous, attains its maximum on H. In order to show uniqueness, suppose that f attains its maximum at two distinct points. Then, it remains to show that the maximum is attained only once by contradiction. Assume, f is attained at two different points, (u_1, v_1) and (u_2, v_2). Let

$$M = \max(f(u, v)) = f(u_1, v_1) = f(u_2, v_2).$$

Either $u_1 > u_2, \;\; v_1 < v_2$, or $u_1 < u_2, \;\; v_1 > v_2$. Let's assume the first possibility. By the convexity of K, $(u_3, v_3) = (1/2)(u_1, v_1) + (1/2)(u_2, v_2) \in K$. Compute

$$\begin{aligned}
f(u_3, v_3) &= (\frac{u_1 + u_2}{2} - u_0)(\frac{v_1 + v_2}{2} - v_0) \\
&= (\frac{u_1 - u_0}{2} + \frac{u_2 - u_0}{2})(\frac{v_1 - v_0}{2} + \frac{v_2 - v_0}{2}) \\
&= (1/4)[(u_1 - u_0)(v_1 - v_0) + (u_2 - u_0)(v_2 - v_0)] \\
&\quad + (u_1 - u_0)(v_2 - v_0) + (u_2 - u_0)(v_1 - v_0)] \\
&= (1/4)[2M + 2M + (u_1 - u_0)(v_2 - v_0) \\
&\quad + (u_2 - u_0)(v_1 - v_0) - (u_1 - u_0)(v_1 - v_0) \\
&\quad - (u_2 - u_0)(v_2 - v_0)] \\
&= M + (1/4)[(v_2 - v_0)(u_1 - u_2) + (v_1 - v_0)(u_2 - u_1)] \\
&= M + (1/4)[(u_1 - u_2)(v_2 - v_1)] \\
&> M
\end{aligned}$$

This is a contradiction since M is the maximum. Similarly, the second possibility is analogous. Thus, f attains at only one point.

□

Having proved the Lemma, let's prove the Nash Theorem stating that **there is a unique arbitration procedure Ψ satisfying Nash's axioms.**

Proof. (Morris, 139) Construction of Ψ
Case (1): There is some pair of payoffs (u, v) in the cooperative region K that is greater that the status quo point such that $u_0 \leq u$ and $v_0 \leq v$. Let H be the set of all such points (u, v). Using the previous Lemma, define.

$$f(u, v) = (u - u_0)(v - v_0) \text{ for } (u, v) \in H.$$

We need to show that there exists some unique pair of payoffs (u^*, v^*) that the function $f(u, v)$ attains its maximum value. Define

$$\Psi(L, (u_0, v_0)) = (u^*, v^*)$$

Case(2): There does not exist a pair off payoffs $(u, v) \in K$ greater than the status quo point such that $u > u_0$ and $v > v_0$. This case can divided into following subcases.

Case(2a): There is (u_0, v) such status quo payoff u_0 for the first player is greater than any other payoff u, But, there is another payoff for the second player v such that $v > v_0$. Then, $\Psi(K, (u_0, v_0)) = (u_0, v)$.

Case(2b): Similarly, there is (u, v_0) such that $u > u_0$ and $\Psi(K, (u_0, v_0)) = (u, v_0)$.

Case(2c): There does not exist any other pair of payoffs $(u, v) \in K$ that would be greater than the status quo point. Then, $\Psi(K, (u_0, v_0)) = (u_0, v_0)$.

We notice that Case(2a) and Case(2b) are actually the same. Note that Case(2a) and Case(2b) cannot both be true. Assume that the Case (2) is true and Case (1) is false. Define

$$(u', v') = \tfrac{1}{2}(u_0, v) + \tfrac{1}{2}(u, v_0).$$

By convexity, $(u', v') \in K$ and satisfies the condition of Case(1). This is a a contradiction. Now, let's verify Nash axioms.

Axioms 1 and 3: It can be noticed that axioms 1 and 3 apparently hold in all cases. The arbitration pair has to be in the feasible region and has to be larger than the pair of individual maximum values.

Axiom 2 (Pareto Optimality): By contradiction, assume that there is $(u, v) \in K$ that dominates (u^*, v^*) and $(u, v) \neq (u^*, v^*)$. Considering Case (1), it would be $(u - u_0) \geq (u^* - u_0)$, $(v - v_0) \geq (v^* - v_0)$ and at least one of these inequalities would be strict because (u, v) is different from (u^*, v^*). Therefore, $f(u, v) \geq f(u^*, v^*)$. This contradicts the construction of (u^*, v^*) since f attains its maximum at (u^*, v^*). Case (2a): $u^* = u_0 = u$. Then, Case (2b) does not hold since one of them can be true. Then, $v > v^*$ that contradicts the definition of v^*. Analogously for Case (2b). Whereas in Case

(2c): $(u^*, v^*) = (u_0, v_0)$; if $u > u_0$, then Case (2b) is true, for $v > v_0$, it would be the Case (2a). Again, we have a contradiction.

Axiom 4 (Independence of Irrelevant Alternatives): If K, the payoff region, contains K' and both pairs (u_0, v_0) and (u^*, v^*) are in K', then $\Psi(K', (u_0, v_0)) = (u^*, v^*)$. In Case (1), the maximum value of the function f over $H \cap K'$ is less than or equal to its maximum over H. Because $(u^*, v^*) \in K'$, the two maximas are equal. Hence, $\Psi(P', (u_0, v_0)) = \Psi(P, (u_0, v_0)$. Similarly, it would be for other cases.

Axiom 5 (Independence under Linear Transformations): Assume that K' is derived from K by the linear transformation

$$u' = au + b,\ v' = cv + d, \text{ where } a, c > 0$$
$$\text{So, } \Psi(K', (au_0 + b, cv_0 + d)) = (au^* + b, cv^* + d).$$

Regarding Case (1), it also holds for a payoff region K' with the status quo $(au_0 + b, cv_0 + d)$. Therefore,

$$(u' - (au_0 + b))(v' - (cv_0 + d)) = ac(u + u_0)(v + v_0).$$

Because $a, c > 0$, maximum value of the function on the left side is obtained at $(au^* + b, cv^* + d)$. Therefore, the axiom 5 is true for Case (1). Similarly, it would be for other cases.

Axiom 6 (Symmetry): Assume $u^* = v^*$. It does not matter how we label the pair of payoffs. By Symmetry, $(v^*, u^*) \in K$. For Case (1), we get $f(u^*, v^*) = f(v^*, u^*)$. So, f attains its maximum at two points. This is a contradiction since f obtains its maximum at a unique point by the previously introduced Lemma. Due to symmetry, in Cases (2a) and (2b), if one held, the latter would hold as well. But, both of the cases cannot be true. Hence, there is a contradiction. Similarly for Case (2c).

Thus, all the axioms are satisfied. Now, we need to show the uniqueness of Ψ using contradiction. Assume there is a different arbitration procedure Ψ_1 satisfying the abovementioned axioms. Holding the same payoff region K and status quo point $(u_0, v_0) \in K$, assume that $\Psi \neq \Psi_1$. Then,

$$(u_1, v_1) = \Psi_1(K, (u_0, v_0)) \neq \Psi(K, (u_0, v_0)) = (u^*, v^*).$$

Assuming the validity of Case (1). Then, $u^* > u_0$ and $v^* > v_0$. Denote

$$u' = \frac{u-u_0}{u^*-u_0},\ v' = \frac{v-v_0}{v^*-v_0}$$

This linear change of variables takes (u_0, v_0) into $(0, 0)$ and (u^*, v^*) into $(1, 1)$. Hence, by the fifth Axiom,

$$\Psi(K', (0, 0)) = (1, 1).$$

Also, by the same Axiom,

$$\bar{\Psi}(K', (0, 0)) \neq (1, 1)$$

Now, if $(u', v') \in K'$, then

$$u' + v' \leq 2$$

Now, assume $u' + v' > 2$. By the convexity of K',

$$t(u', v') + (1 - t)(1, 1) \in K,\ 0 \leq t \leq 1$$

Now define, for $1 \leq t \leq 1$,

$$h(t) = f(t(u', v') + ((1 - t)(1, 1)) = (tu' + (1 - t))(tv' + (1 - t)).$$

Then $h(0) = 1$ and the derivative of $h(t)$ is

$$h'(t) = 2tu'v' + (1 - 2t)(u' + v') - 2(1 - t)$$

Plugging 0 into $h'(t)$, we will have

$$h'(0) = u' + v' - 2 > 0$$

Hence, there is a small positive t such that $h(t) > 1$. However, this a contradiction to the definition of (u^*, v^*), since $f(1, 1) = 1$. Now, let $\hat{K}$ be the symmetric convex hull of K'. By Lemma 5.3 [Let K be a subset of $\mathbb{R}^2$ and k be a number such that $u + v \leq k$ for every point $(u, v) \in K$], $s + t \leq 2$ for all $(s, t) \in \hat{K}$ and thus if $(a, a) \in \hat{K}$, then $a \leq 1$. Because $\hat{K}$ is symmetric, by Axiom 6

$$\bar{\Psi}(\hat{K}, (0, 0)) = (1, 1).$$

Otherwise, there is a point $(a, a) \in \hat{K}$ with $a > 1$. However, according to Axiom 4,

$$\bar{\Psi}(K', (0, 0)) = (1, 1)$$

This is a contradiction that shows uniqueness in Case (1). Cases (2a,2b,2c) can be understood by Axiom 1. From Axiom 1, $\bar{u} = u_0 = u^*$. Because both (u^*, v^*) and $(\bar{u}, \bar{v})$ are Pareto optimal, $u^* = \bar{u}$ and $v^* = \bar{v}$. This is a contradiction to the supposition that (u^*, v^*) and $(\bar{u}, \bar{v})$ are distinct.

□

8 Tom's and Chris' Arbitration Pair

Since we now know the meanings of the Nash Bargaining Axioms and Theorem, we can apply them to find the arbitration pair of Tom and Chris. We know that:

- The status quo for Tom and Chris is the pair of payoffs $(0, 1/2)$,
- The bargaining set would be the line segment from $(1, 5)$ to $(5, 1)$, which gives the line segment $v = -u + 6$ using the linear equation formula

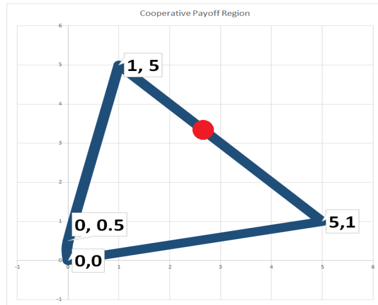
Using the above information we can plug into the $f(u, v)$ function. Then,

$$f(u, v) = (u - u_0)(v - v_0) = (u - 0)(-u + 6 - 1/2) = -u^2 + 11/2u$$

Notice that $f(u, v)$ can be written now in terms of u. So, $f(u) = -u^2+11/2u$. This is just a quadratic function which is concave down. So, $f(u)$ attains its maximum at its vertex. Using calculus, $f'(u) = 0$. Then,

$$f'(u) = -2u + 11/2 = 0$$
$$u^* = 2.75 \text{ and } v^* = 3.25$$

Thus, the $f(u, v)$ attains its maximum when $u^* = 2.75$ and $v^* = 3.25$. Hence, the point $(2.75, 3.25)$ is the arbitration pair from the bargaining set produced by the arbitration procedure where 2.75 is the happiness score for Tom and 3.25 is the happiness score for Chris. So how can Tom and Chris achieve these pair of payoffs? Let's plot this point on graph of the cooperative payoff region for Tom and Chris.



Note that the point $(2.75, 3.25)$ is much closer to the point $(1, 5)$ than $(5, 1)$. In addition, recall that the pair of payoffs $(1, 5)$ is when Tom and Chris agree to watch Shark Tank together whereas $(5, 1)$ is when they agree on watching together Economics News. This means that they should watch Shark Tank together more often in order to generate pair of payoffs $(2.75, 3.25)$. It turns out that they should watch together Economics News $7/16$ of the time and Shark Tank $9/16$ of the time in order to generate above-mentioned happiness score.

9 Applications

The two-person bargaining problem can be applied to a situation between a buyer and a seller. The negotiation between them would be regarding the price of a product or a service that the buyer would like buy at and the seller would like to sell at. They need to agree to agree on a certain price so that the transaction, which is a cooperation in this case, can done and they would be satisfied with.

Of course, the bargaining problem is not restricted to two players. It can be generalized to $n-$person cooperative game where n is the number of players in a game. Imagine, there are two classes. Each class consists of 10 students. These two classes like studying in the Math room. Because the room's capacity is limited, only one class can be in there at a time. Now, suppose that these two classes decide to negotiate together how often each class can spend their time studying in there. This time, we have 20 people in the negotiation. But, if each class acts as one body, then 20 people are reduced to players actually. This is an assumption if ten people from each class have the same amount of payoffs and way of thinking. If each person in class has a different payoff and thinks differently, the situation would show more than two conflicting interests. So, there would be more than two players.

Another two-person bargaining situation can be seen in the economic oligopolistic settings. Oligopolies are pervasive around the world and have growing rapidly. Compared with a monopoly wherein one company dominates a certain market, an oligopoly consists of minimum two companies having significant impact on an industry. The two oligopolistic companies in the same industry might negotiate about the price of a certain similar product that they would charge customers or the salary to give to their employees. One of the reasons why the oligopolistic companies would like to cooperate with one another is because they do not want to create a competition. By cooperation, they would have a larger market share in the specific industry and make it harder for other companies to enter. Recall that the above solutions are according to the assumptions previously made. As the result of different assumptions, a different model might be created. Therefore, there would be a different solution.

In conclusion, the Nash Bargaining Axioms, Theorem and other abovementioned concepts give us a tool to solve the two-person bargaining problem. However, the process to find a solution is heavily based on assumptions that we make. Since assumptions are discretionary, differently stated assumptions produce different models and solutions. As long as we prudently state our assumptions, the model and the solution should be reasonable.

References

[1] Mayberry, John P. *Game-theoretic Models of Cooperation and Conflict.*Boulder: Westview, 1992. Print.

[2] Morris, Peter. *Introduction to Game Theory*. New York: Springer-Verlag, 1994. Print.

[3] http://www.investopedia.com/ask/answers/121514/what-are-some-current-examples-oligopolies.asp

[4] http://www.rijpm.com/pre˙reading˙files/Open˙Options˙Introduction˙To˙Game˙Theory.pdf