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On Perfect Numbers

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Abstract

A perfect number is a number that is equal to the sum of its divisors excluding itself. Theorems proven by Euclid and Euler show that finding a new Mersenne prime is equivalent to the discovery of a new even perfect number. Even perfect numbers have a systematic way of being found and have various ways in which they can be characterized, but the same is not true for odd perfect numbers. In fact, the existence of an odd perfect number has not been proven or disproven. This paper discusses the categorizations of even perfect numbers and possible characteristics of odd perfect numbers.

1 Introduction

A **perfect number** is a positive integer that is equal to the sum of its positive divisors excluding itself. The ancient Greeks were well versed in the existence of perfect numbers and held those that they were aware of (the first four) in high regard. Perfect numbers were thought to have mystical healing properties and were even used to illustrate the existence of God. Examples of perfect numbers include $6, 28, 496$, and 8128 because:

$$6 = 1 + 2 + 3$$

$$28 = 1 + 2 + 4 + 7 + 14$$

$$496 = 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248.$$

Such examples could entice the notion that perfect numbers are found frequently and are relatively close to one another, but this could not possibly be further from the truth. As of the writing of this paper, there are 49 known perfect numbers, with the newest one having been discovered in early January 2016. This newest perfect number number is

$$2^{(74,207,280)}(2^{74,207,281} - 1)$$

and is $44,677,235$ digits long. Large primes and perfect numbers such as these are found by many people who cooperate together as part of the Great Internet Mersenne Prime Search (GIMPS). While finding new perfect numbers is certainly exciting, mathematicians are more concerned with finding a systematic and reliable *method* of finding perfect numbers. Perfect numbers have continued to be of interest for thousands of years because they have been studied by some of the brightest mathematicians in history, but have yet to reveal the entirety of their nature. This paper seeks to explain the progress made in the understanding of perfect numbers and to discuss why perfect numbers continue to be one of the world's oldest unsolved puzzles.

2 Euclid's Perfect Number Theorem

Born in Alexandria around 330 B.C, the great mathematician Euclid was instrumental in the advances made in the study of perfect numbers. In Book

IX Proposition 36 of his famous work, *Elements*, he states the following: "If as many numbers as we please beginning from a unit are set out continuously in double proportion until the sum of all becomes prime, and if the sum multiplied into the last makes some number, then the product is perfect." The result of Euclid's studies of perfect numbers is Euclid's Perfect Number Theorem. Euclid's Theorem is widely considered to be the first step mankind took to understanding the nature of perfect numbers [9].

Theorem 1 (Euclid's Perfect Number Theorem)**.** *If* $2^p - 1$ *is a prime number, then* $2^{p-1}(2^p - 1)$ *is a perfect number.*

Proof. Let p be an integer such that $2^p - 1$ is a prime number. We aim to show that $2^{p-1}(2^p - 1)$ is a perfect number. Now let $q = 2^p - 1$, so that $2^{p-1}(2^p - 1) = 2^{p-1}q$. We can write out the divisors of 2^{p-1} as follows: $1, 2, 4, 8, 16, \ldots, 2^{p-1}$. Then, we can write out the other divisors of $2^{p-1}q$ as follows: $q, 2q, 4q, 8q, 16q, \ldots, 2^{p-2}q$. We proceed by adding $1, 2, 4, 8, 16, \ldots, 2^{p-1}$ first. Recall that

$$1 + x + x^2 + x^3 + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1}.$$

In this case, $n = p$ and $x = 2$ so

$$1 + 2 + 4 + 8 + 16 + \cdots + 2^{p-1} = \frac{2^p - 1}{2 - 1} = 2^p - 1 = q \qquad (1)$$

Next, we can use the same formula to find $q + 2q + 4q + 8q + 16q + \cdots + 2^{p-2}q$.

It is clear that

$$q + 2q + 4q + 8q + 16q + \cdots + 2^{p-2}q = q(1 + 2^1 + 2^2 + 2^3 + \cdots + 2^{p-2})$$

so now $x = 2$ and $n = p - 1$. This gives us:

$$\begin{aligned} q(1 + 2^1 + 2^2 + 2^3 + \cdots + 2^{p-2}) &= q(\frac{2^{p-1} - 1}{2 - 1}) \\ &= q(2^{p-1} - 1) \qquad (2) \end{aligned}$$

Finally, we can sum all the divisors of $2^{p-1}q$ by combining (1) and (2):

$$\begin{aligned} &1 + 2 + 4 + 8 + 16 + \cdots + 2^{p-1} + q + 2q + 4q + 8q + 16q + \ldots 2^{p-2}q \\ &= q + q(2^{p-1} - 1) \\ &= q + q(2^{p-1}) - q \\ &= 2^{p-1}q \\ &= 2^{p-1}(2^p - 1) \end{aligned}$$

So we have shown that if $q = 2^p - 1$ is prime, then $2^{p-1}(2^p - 1)$ is perfect. $\square$

3 Euler's Perfect Number Theorem

Naturally, the question that arises after proving Euclid's Perfect Number Theorem is whether it describes all perfect numbers. That is to say, are all perfect numbers of the form $2^{p-1}(2^p - 1)$? The next significant step in the answering of this question and the understanding of perfect numbers was made two thousand years after Euclid's results by Swiss mathematician Leonhard Euler. Euler proved that Euclid's formula for perfect numbers holds true for all even perfect numbers.

Theorem 2 (Euler's Perfect Number Theorem)**.** *If n is an even perfect number, then it is of the form*

$$n = 2^{p-1}(2^p - 1)$$

where p is some prime and $2^p - 1$ is a Mersenne prime.

A **Mersenne prime** is a prime of the form $2^p - 1$. Euler's theorem is indicates that there is a one-to-one relationship between Mersenne primes and even perfect numbers, so it is of significant importance that we prove this theorem.

3.1 The Sigma Function

Before Euler's Perfect Number Theorem can be proved, it is imperative that we define and study the properties of what is known as the sum of divisors function. We define the sum of divisors function σ as the following:

$$\sigma(n) = \text{the sum of all unique divisors of } n \text{ including 1 and } n.$$

Examples include

$$\sigma(4) = 1 + 2 + 4 = 7$$

$$\sigma(6) = 1 + 2 + 3 + 6 = 12$$

$$\sigma(18) = 1 + 2 + 3 + 6 + 9 + 18 = 39$$

Note that n is perfect when $\sigma(n) = 2n$. If we are to prove Euler's theorem, then naturally we must show that $\sigma(2^{p-1}(2^p - 1)) = 2(2^{p-1}(2^p - 1))$. Before we address such a proof we must prove the following:

Theorem 3 (Properties of the Sigma Function)**.**
(a) If p is a prime and $k \geq 1$, then

$$\sigma(p^k) = 1 + p + p^2 + \cdots + p^k = \frac{p^{k+1} - 1}{p - 1}.$$

(b) If gcd$(m, n) = 1$, then

$$\sigma(mn) = \sigma(m)\sigma(n).$$

We begin by proving statement (a).

Proof. If p is prime, then it is divisible only by 1 and itself. It follows that the divisors of p^k are of the form p^i for all i such that $i \leq k$. That is to say, we can write $\sigma(p^k) = 1 + p + p^2 + \cdots + p^{k-1} + p^k$. Note that this sum can be written as having the form $\sum_{j=0}^{n} ar^j$. We begin by defining S such that

$$S = \sum_{j=0}^{n} ar^j$$

and proceed by multiplying S by r:

$$\begin{aligned} rS &= r\sum_{j=0}^{n} ar^j \\ &= \sum_{j=0}^{n} ar^{j+1} \\ &= \sum_{k=1}^{n+1} ar^k \\ &= \sum_{k=0}^{n} ar^k + (ar^{n+1} - a) \\ &= S + (ar^{n+1} - a). \end{aligned}$$

Solving for S gives us that

$$S = \frac{ar^{n+1} - a}{r - 1}$$

But in our case $a = 1$ and $r = p$, thus $\sigma(p^k) = \frac{p^{k+1}-1}{p-1}$ □

To prove property (b), we must first prove the following lemma [8].

Lemma 1. *Suppose a and b are relatively prime integers with $\{a_i : 1 \leq i \leq s\}$ and $\{b_j : 1 \leq j \leq t\}$ being all the divisors of a and b respectively. Then*

$S = \{a_i b_j : 1 \leq i \leq s, 1 \leq j \leq t\}$ *are all the divisors of* ab.

Proof. It is clear that each element of the form $a_i b_j$ is a divisor of ab. So it follows that S contains only divisors of ab. We aim to show that all of the divisors of ab are present in S.

Suppose $d \mid ab$ and let $D = \gcd(a, b)$ and $d' = \frac{d}{D} \in \mathbb{Z}$. Because $D \mid a$, $D = a_j$ for some j. It follows that $d = Dd' = a_j d'$. We must show d' divides b.

Because $d \mid ab$, $dk = ab$ for some integer k, so $Dd'k = ab$ if $d'k = \frac{a}{D}b$.

Because $\gcd(a, b) = D$, we have that $\gcd(\frac{a}{d}, \frac{b}{D}) = 1$, so $\gcd(\frac{a}{d}, d') = 1$. Thus, by Euclid's Lemma $d' \mid b$ so $d' = b_i$ for some i as desired. □

Now we can proceed with proving property (b).

Proof. Using the same notation as above, let $\{a_i : 1 \leq i \leq s\}$ and $\{b_i : 1 \leq j \leq t\}$ be the divisors of a and b respectively. Then we have $\sigma(a)\sigma(b) = (\sum_i a_i)(\sum_j b_j) = \sum_i \sum_j a_i b_j = \sum_{d \mid ab} d = \sigma(ab)$. □

3.2 Proving Euler's Perfect Number Theorem

Now we use our newly found knowledge of $\sigma(n)$ to prove Euler's theorem. Recall that Euler's Perfect Number Theorem states the following: if n is an even perfect number, then it is of the form

$$n = 2^{p-1}(2^p - 1)$$

where $2^p - 1$ is a Mersenne prime [11].

Proof. Let n be an even perfect number, then we can write it as $n = 2^{k-1}m$ where m is some odd integer. It follows from property (b) of the σ function that since $\gcd(2^{k-1}, m) = 1$, $\sigma(2^{k-1}m) = \sigma(2^{k-1})\sigma(m)$. Furthermore, Theorem 3(a), we obtain $\sigma(2^{k-1}m) = (2^{k-1})\sigma(m)$. Recall that if n is a perfect number, $\sigma(n) = 2n$ so we have that

$$\sigma(n) = 2n = 2^{k-1}m = (2^k - 1)\sigma(m).$$

Let $s = \sigma(m) - m$; that is to say, s is the sum of all the proper divisors of m that are less than m. Now we have $\sigma(n) = 2^k m = (2^k - 1)(m + s)$. Distributing gives us $2^k m = 2^k m - m + (2^k - 1)s$. Subtracting $2^k m$ from both sides gives us that $m = (2^k - 1)s$.

This implies that $s \mid m$ and $s < m$. However, recall that s is the sum of all the proper divisors of m that are less than m. If $s \mid m$, then s must be one of these divisors. The only way this is possible is if $s = 1$. Thus $m = (2^k - 1)$, which means that $n = 2^{k-1}(2^k - 1)$. $\square$

3.3 Characterizing Even Perfect Numbers

With Euclid's and Euler's theorems proven, mathematicians have a reliable and systematic way of finding the next even perfect number provided that a new Mersenne prime has been found. The next natural step would be to characterize the even perfect numbers that we know exist so that we know the properties that future even perfect numbers must have. That is to say, now that we know what the next even perfect number will look like (even though we might not know exactly what it is) it is only natural to identify patterns exhibited by even perfect numbers and prove that they hold for all even perfect numbers. For the sake of inquiry and convenience, a table of the first few even perfect numbers is produced below.

Rank	Perfect Number	Digits	Year
1	6	1	4th Century BC
2	28	2	4th Century BC
3	496	3	4th Century BC
4	8128	4	4th Century BC
5	33550336	8	1456
6	8589869056	10	1588
7	137438691328	12	1588
8	2305843008139952128	19	1772
9	26584559...613842176	37	1883
10	191561942...548169216	54	1911
11	131640364...783728128	65	1914
12	144740111...199152128	77	1876
13	235627234...555646976	314	1952
⋮	⋮	⋮	⋮
49	451129962...930315776	44,677,235	2016

It is clearly observable that the fourteen even perfect numbers shown in the table end in either 6 or 8. It can be verified using a full version of this table [2] that this trend continues to hold for every discovered even perfect number. We prove that this characteristic is a property of all even perfect numbers [10].

Theorem 4. *All even perfect numbers greater than* 6 *end in with either* 6 *or* 8*.*

Proof. Let N be an even perfect number. Recall that by Theorem 2, if N is an even perfect number, then $N = 2^{n-1}(2^n - 1)$ where n is a positive prime integer. It is true that every prime greater than 2 is of the form $4m + 1$ or $4m + 3$.

Now consider the case where n is of the form $4m + 1$. If n is of the form $4m + 1$. It follows that

$$N = 2^{4m}(2^{4m+1} - 1) = 16^m(2 \cdot 16^m - 1).$$

Finding the first digit of this integer is equivalent to finding what this integer is modulo 10. Using properties of modular arithmetic, we see that $16^m(2 \cdot 16^m - 1) \equiv (6^m)(2 \cdot 6^m - 1)$. We can simplify further to find that $(6^m)(2 \cdot 6^m - 1) \equiv 6(12 - 1) \equiv 6 \pmod{10}$. Thus if n is of the form $4m + 1$, N ends in 6.

Next, consider the case where n is of the form $4m + 3$. Then the following is true:

$$N = 2^{4m+2}(2^{4m+3} - 1) = [(2^{4m})(2^2)][(2^{4m}(2^3) - 1)].$$

Simplifying gives us that $N = 4 \cdot 16^m(8 \cdot 16^m - 1)$. Similar to the case above, finding what this number is modulo 10 is equivalent to finding the final digit of this number. Using properties of modular arithmetic gives us that $N = 4 \cdot 16^m(8 \cdot 16^m - 1) \equiv 4 \cdot 6(8 \cdot 6 - 1) \equiv 4(8 - 1) \equiv 8 \pmod{10}$. Thus, if n is of the form $4m + 3$, the final digit of N is 8. □

Next we follow up by proving an observation that is not as obvious and also presented in [10].

Theorem 5. *Every even perfect number greater that* 6 *can be expressed as the sum of the cubes of consecutive odd integers.*

Proof. Recall that $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$.
Let $m = 2^{(p-1)/2}$ where p is prime. We want to rewrite the sum of the cubes of consecutive odd integers so that it is of the form $2^{p-1}(2^p - 1)$. That is to say, we claim that:

$$1^3 + 3^3 + \cdots + (2m - 1)^3 = 2^{p-1}(2^p - 1)$$

Note that the sum of the cubes of consecutive odd positive integers less than $2m$ is equal to the difference of the sum of the cubes of all positive integers less than $2m$ and the sum of the cubes of all even positive integers less than

$2m$. So we can say that:

$$\begin{aligned}
1^3 + 3^3 + \cdots + (2m-1)^3 &= (1^3 + 2^3 + \cdots + (2m)^3) - (2^3 + 4^3 + \cdots + (2m)^3) \\
&= \frac{(2m)^2(2m+1)^2}{4} - 2^3\frac{m^2(m+1)^2}{4} \\
&= m^2(2m+1)^2 - 2m^2(m+1)^2 \\
&= m^2(4m^2 + 4m + 1 - 2m^2 - 4m - 2) \\
&= m^2(2m^2 - 1).
\end{aligned}$$

Substituting for m gives us $(2^{(p-1)/2})^2(2(2^{(p-1)/2})^2 - 1)$, or simply $2^{p-1}(2^p - 1)$ as desired. □

It is important in mathematics that conjectures or general observances should not be treated as theorems until they can be proven. For example, if one were charged with finding the first one hundred abundant numbers (numbers for which $\sigma(n) > 2n$), one might be inclined to assume that only even numbers could be abundant. However, it is only necessary to provide a counterexample to debunk this conjecture. Surely enough, there is a counterexample: 945 is abundant and odd. Similarly, just because we have yet to find an odd perfect number does not give us sufficient cause to believe that there must not be any. We must *prove* that there can be no such thing as an odd perfect number. Interestingly enough, we have yet to prove that there does not exist an odd perfect number, but mathematicians have been able to successfully prove characteristics an odd perfect number must have if it were to exist.

4 Determining the Existence or Nonexistence of Odd Perfect Numbers

Now that we have discussed methods for which even perfect numbers can be produced, it is only natural to ask the questions "Are there odd perfect numbers?" and "What methods are there for finding odd perfect numbers?" To this day, mathematicians have been unsuccessful in finding an odd perfect number or to prove that there is no such thing as an odd perfect number. This is partly what makes the study of perfect numbers so interesting: even with the greatest mathematical minds and computing power that mankind has accumulated over the course of thousands of years, there exists a simple question about to the integers that remains unanswered. It is only natural that mankind has been so insistent on the pursuit of knowledge of odd perfect numbers.

4.1 Possible Characteristics of Odd Perfect Numbers

Lemma 2. *If n is of the form $6k-1$, then n is not perfect.* [5]

Proof. We begin by assuming that n is a positive integer of the form $6k-1$. Then $n \equiv -1 \pmod 3$. Now suppose that d is a divisor of n. This means $n = d \cdot \frac{n}{d}$, so we can write $n = d \cdot \frac{n}{d} \equiv -1 \pmod 3$. This implies two cases

(1) $d \equiv -1 \pmod 3$ and $\frac{n}{d} \equiv 1 \pmod 3$ or (2) $d \equiv 1 \pmod 3$ and $\frac{n}{d} \equiv -1 \pmod 3$. Using either case, we have that $d + \frac{n}{d} \equiv 0 \pmod 3$. This means that

$$\sigma(n) = \sum_{d|n, d<\sqrt{n}} d + \frac{n}{d} \equiv 0 \pmod 3.$$

We can directly compute $2n = 2(6k-1) = 12k - 2 \equiv 1 \pmod 3$. Thus n cannot be perfect. $\square$

Not only did Euler provide us with a useful theorem for even perfect numbers, he also provided us with an equally impactful theorem that allowed us to study the form of odd perfect numbers [6].

Theorem 6 (Euler's Odd Perfect Number Theorem)**.** *Any odd perfect number n must be of the form $n = p^\alpha m^2$ with p prime and $p \equiv \alpha \equiv 1 \pmod 4$ This implies that $n \equiv 1 \pmod 4$.*

Proof. Let n be an odd perfect number, write $n = a_1^{e_1} a_2^{e_2} a_3^{e_3} \ldots a_r^{e_r}$ where a_i is prime. Because n is odd, it must be true that all a_i are odd as well. Recall that we use the fact that n is perfect to write that $\sigma(n) = 2n$. Since

$$\sigma(n) = \sigma(a_1^{e_1} a_2^{e_2} a_3^{e_3} \ldots a_r^{e_r})$$

then it must be true that $\sigma(n) = \sigma(a_1^{e_1})\sigma(a_2^{e_2})\sigma(a_3^{e_3}) \cdots \sigma(a_r^{e_r})$. Note that for some a^e, $\sigma(a^e) = 1 + a + a^2 + a^3 + \cdots + a^e$ is the sum of $e+1$ terms, so $\sigma(a^e)$

is odd only when e is even. Since

$$\sigma(a_1^{e_1}a_2^{e_2}a_3^{e_3}\cdots a_r^{e_r}) = \sigma(a_1^{e_1})\sigma(a_2^{e_2})\sigma(a_3^{e_3})\cdots\sigma(a_r^{e_r}) = 2a_1^{e_1}a_2^{e_2}a_3^{e_3}\cdots a_r^{e_r}$$

$\sigma(n)$ is limited to having at most one factor of two. Thus, all e_i are even except for one. Without loss of generality, let e_1 odd. This means $n = a_1^{e_1}a_2^{2f_1}a_3^{2f_2}a_4^{2f_3}\ldots a_r^{2f_{r-1}}$, where $e_{i+1} = 2f_i$. We have that $2 \mid \sigma(a_1^{e_1})$ but $4 \nmid \sigma(a_1^{e_1})$. Since a_1 is odd, either $a_1 \equiv 1 \pmod 4$ or $a_1 \equiv -1 \pmod 4$. If the latter is true then

$$\sigma(a_1^{e_1}) = 1 + a_1 + a_1^2 + \cdots + a_1^e$$
$$\equiv 1 + (-1) + 1 + (-1) + 1 + \cdots + (-1) + 1 \equiv 0 \pmod 4$$

This is a contradiction because we have established that $4 \nmid \sigma(a_1^{e_1})$. Thus, it must be true that $a_1 \equiv 1 \pmod 4$. Thus

$$\sigma(a_1^{e_1}) = 1 + a_1 + a_1^2 + a_1^3 + \cdots + a_1^e$$
$$\equiv 1 + 1 + 1 + \cdots + 1 + 1 \equiv e_1 + 1 \pmod 4$$

Because e_1 is odd, $e_1 + 1 \equiv 0 \pmod 4$ or $e_1 + 1 \equiv 2 \pmod 4$. But the former cannot be true because this would mean $4 \mid \sigma(a_1^{e_1})$. This leaves us with the statement $e_1 + 1 \equiv 2 \pmod 4$ if and only if $e_1 + 1 = 4e + 1$. Consequently, $N = a_1^{e_1}a_2^{2f_1}a_3^{2f_2}a_4^{2f_3}\ldots a_r^{2f_r-1}$, for $e_1 \equiv 1 \pmod 4$. □

Euclid's Odd Perfect Number Theorem was a large step towards understanding odd perfect numbers. The French mathematician Jacques Touchard made the next leap in the understanding of odd perfect numbers by proving the following about the form they must take [5].

Theorem 7 (Touchard's Theorem)**.** *Any odd perfect number must have the form* $12m+1$ *or* $36m+9$*.*

Proof. Let n be an odd perfect number. Applying Lemma 1, we know that n cannot be of the form $6k-1$, so it must be of the form $6k+1$ or $6k+3$. By Euler's Odd Perfect Number Theorem, all odd perfect numbers are congruent to $1 \pmod 4$, so either $n \equiv 1 \pmod 6$ and $n \equiv 1 \pmod 4$ or $n \equiv 3 \pmod 6$ and $n \equiv 1 \pmod 4$. Using these two relations, we can reason that n must be of the form $12m+1$ or $12m+9$. In the case that $n = 12m+9$ and $3 \nmid m$, it must be true that

$$\sigma(n) = \sigma(12m+9) = \sigma(3(4m+3)) = \sigma(3)\sigma(4m+3) = 4\sigma(4m+3)$$

This would mean that n is a multiple of four so $\sigma(n) \equiv 0 \pmod 4$. This means $2n = 2(12m+9) \equiv 2 \pmod 4$ so n cannot be perfect. Thus, it must be true that $3 \mid m$ so n has the form $n = 36m+9$. □

4.2 On Going Fields of Study

The work to characterize and categorize odd perfect numbers is of great importance; however, it is of equal importance that we prove or disprove the existence of an odd perfect number. Disproving nonexistence can be done in the form of an intricate proof or by simply providing a counterexample. However, the case of disproving nonexistence by counterexample remains elusive. Nevertheless, significant progress has been made by mathematicians such as Brent and Cohen, whose work revolves around placing a lower bound on the first odd perfect number [1]. Below is a table showcasing the progress made thus far in placing a lower bound on the first odd perfect number [4]. Similar topics that are currently worked on include placing a lower bound on the number of unique prime divisors an odd perfect number and placing upper bounds on the two smallest prime divisors of an odd perfect number [7].

Author	Bound
Kanold (1957)	10^{20}
Tuckerman (1973)	10^{36}
Hagis (1973)	10^{50}
Brent and Cohen (1989)	10^{160}
Brent and Co. (1991)	10^{300}

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