Prime Numbers Arising from Quadratic Polynomials

Avineet Pannu

September 17, 2012

Abstract

For $f(1), f(2), ..., f(n)$, define $\mathcal{S}$ (a,b,n) to be the smallest possible integer m, such that $f(1), f(2), ..., f(n)$ are all distinct $\pmod{m}$. Zhi-Wei Sun claims in his paper *A Simple Way to Generate All Primes* that he is able to generate exactly all primes using the quadratic $f(t) = t(t-1)$ and modulo arithmetic defined above. We used the function $\mathcal{S}$ (a,b,n) and defined it using the coefficients of a quadratic in the form of $f(t) = t(at+b)$ and n just being the term in the sequence. Continuing off of Zhi-Wei Sun's research which looked at $\mathcal{S}$(1,-1,n), we explore the general case where b is left as a constant, $\mathcal{S}$(1,b,n). Depending on the two different cases where b is either even or odd, we see that $\mathcal{S}$(1,b,n) is a combination of $2p, 2^h$, and p. After that, we show the results of two specific cases, $\mathcal{S}$(2,1,n) and $\mathcal{S}$(3,1,n). Interestingly $\mathcal{S}$(2,1,n) results in 2^h for h such that $n \leq 2^h$ and $\mathcal{S}$(3,1,n) in 3^h for h such that $n \leq 3^h$.

1 Introduction

For a very long time, mathematicians have been trying to find a polynomial that only outputs prime numbers. They have found formulas that at first give prime values, but they eventually fail. For example, $p(n) = n^2 - n + 41$. The first few terms are: $n = 1$ then $p(n) = 41, n = 2$ then $p(n) = 43, n = 3$ then $p(n) = 47, n = 4$ then $p(n) = 53, n = 5$ then $p(n) = 61, n = 6$ then $p(n) = 71, n = 7$ then $p(n) = 83$. However when $n = 41, p(n) = 1681$, which is 41^2, therefore it is not prime. Thus at $n = 41, p(n)$ does not work.

Recently, Zhi-Wei Sun came up with a new idea that uses a function $f(t) = t(t-1)$ and modulo arithmetic. Specifically, for a fixed integer, $n > 1$ let m be the smallest integer such that $f(1), f(2), ..., f(n)$ are distinct modulo m. He claimed that the numbers in which arise this way are always prime. Therefore it is believed that this simple function has a value set of exactly all prime numbers.

Using the same method he explains in his paper *A Simple Way to Generate All Primes*, I used polynomials in the form of $f(t) = at^2 + bt + c$. However, since $f(t_1) \equiv f(t_2) \pmod{m}$ iff $f(t_1) - c \equiv f(t_2) - c \pmod{m}$, we can instead just say $f(t) = at^2 + bt = t(at + b)$. Using this I began to find the first m, that would give me a pairwise incongruent answer when it is $f(t) \pmod{m}$. I called this function $\mathcal{S}$(a,b,n) or an abbreviated $\mathcal{S}$(n); also more commonly referred to as m. In Sun's case his function was $\mathcal{S}$(1,-1,n). In the general case I looked at, I kept b as a variable to investigate how the final answer was different as the value of b changed. This function looked like $\mathcal{S}$(1,b,n). The other two specific cases I looked at were $\mathcal{S}$(2,1,n) and $\mathcal{S}$(3,1,n).

2 Proofs for the general case of $f(t) = t(t + b)$

Theorem 1. *Suppose $n > 1$ is a fixed integer and m will denote an integer such that $f(1), f(2), ..., f(n)$ have distinct remainders $\pmod{m}$, then*
(i) $m \geq 2n + b$
(ii) m is a prime or power of 2 when b is odd

Proof of Theorem 1 (i).
First do the case where $m - b$ is even:

$$\text{Let } x = \frac{m-b}{2} + 1 \text{ and } y = \frac{m-b}{2} - 1.$$

Then using the function $f(t) = t(t + b)$, $f(x) - f(y) = x(x + b) - y(y + b) =$

$$\left(\frac{m-b}{2} + 1\right)\left(\frac{m-b}{2} + 1 + b\right) - \left(\frac{m-b}{2} - 1\right)\left(\frac{m-b}{2} - 1 + b\right).$$

This simplifies to

$$2(m - b) + 2b = 2m \equiv 0 \pmod{2m}.$$

Hence $f(x) \equiv f(y) \pmod{m}$. We call this a collision. Because f(1), f(2),..., f(n) are distinct $\pmod{m}$, we must have $x > n$.

$$x = \frac{m-b}{2} + 1 > n \text{ we easily see that } m \geq 2n + b.$$

Now if $m - b$ is odd:

$$\text{let } x = \frac{m-b+1}{2} \text{ and } y = \frac{m-b-1}{2}.$$

Then using the collision between x and y we obtain $f(x) - f(y) =$

$$\left(\frac{m-b+1}{2}\right)\left(\frac{m-b+1}{2} + b\right) - \left(\frac{m-b-1}{2}\right)\left(\frac{m-b-1}{2} + b\right),$$

which simplifies to

$$\frac{1}{4}(2m - 2b + 2b + 2m - 2b + 2b) = m$$

Again $f(x) \equiv f(y) \pmod{m}$. Therefore using the same previous reasoning $x > n$

$$x = \frac{m-b+1}{2} > n, \text{ which leads to } m \geq 2n + b. \quad \square$$

Lemma 2. *When b is odd, we cannot have $m = 2p$ for p an odd prime.*

Proof of Lemma 2.

So by contradiction, suppose $m = 2p$.

$$\text{Let } x = \frac{p-b}{2} + 1 \text{ and } y = \frac{p-b}{2} - 1.$$

Then using previous methods of collision between x and y we obtain

$$\left(\frac{p-b}{2} + 1\right)\left(\frac{p-b}{2} + 1 + b\right) - \left(\frac{p-b}{2} - 1\right)\left(\frac{p-b}{2} - 1 + b\right),$$

which simplifies to

$$2(p-b) + 2b = 2p \equiv 0 \pmod{2p}.$$

Thus $x > n$ which leads to

$$\frac{p-b}{2} + 1 > n \text{ leading to } 2n + b \leq p = \frac{m}{2}$$

which is impossible since we already established that m itself has to be greater than $2n+b$. Therefore it is not possible that $m/2$ is also greater than $2n+b$. Thus m cannot be 2p. □

Lemma 3. *For any b we cannot have m divisible by p^2 where p is an odd prime.*

Proof of Lemma 3.

By contradiction, assume $p^2 \mid m$

For the case where b is odd,

$$\text{let } x = y + pq \text{ and } y = \frac{p-b}{2}.$$

Then using the same collision as before between x and y we obtain

$$\left(\frac{p-b}{2} + pq\right)\left(\frac{p-b}{2} + pq + b\right) - \left(\frac{p-b}{2}\right)\left(\frac{p-b}{2} + b\right),$$

which simplifies to

$$pq(pq+p) \equiv 0 \pmod{pq}. \text{ Hence } f(x) \equiv f(y) \pmod{m}.$$

Now looking at the case where b is even, let $x = y + pq$ and $y = \frac{2p-b}{2}$. Then using the same collision as before between x and y we obtain

$$\left(\frac{2p-b}{2} + pq\right)\left(\frac{2p-b}{2} + pq + b\right) - \left(\frac{2p-b}{2}\right)\left(\frac{2p-b}{2} + b\right),$$

which simplifies to

$$pq(pq+2p) \equiv 0 \pmod{pq}.$$

Since we have the same collision as before we can say that $x > n$ which once again gives us a contradiction to the property of x and m. □

Proof of theorem 1 (ii).

We know that m is a number such that $f(1), f(2), ..., f(n)$ are distinct (mod m). We know $m \neq 2 * odd$ and $m \nmid p^2$. We want to show that m is a prime or a power of two when b is off. So assume That it is neither of those and therefore $m = pq$ for p is an odd prime and $p \nmid q$

$$\text{Let } x = y + p \text{ and } y = \frac{2}{\gcd(2,q)} - \frac{b+p}{2}.$$

In the case of $2 \mid q$

$$\begin{aligned} x &= \frac{q}{2} - \frac{b}{2} + \frac{p}{2} < \frac{q}{2} + \frac{p}{2} \\ &= \frac{m}{2p} + \frac{m}{2q} \\ &= \frac{m}{2}\left(\frac{1}{p} + \frac{1}{q}\right). \end{aligned}$$

Now take the case that maximizes x to find the upper bound on x. Since $p \nmid q$ and $q > 2$, the values of p and q that would maximize x are p=3 and q=4 which gives us

$$x \leq \frac{m}{2}\left(\frac{1}{3} + \frac{1}{4}\right). \text{ Therefore } x \leq \frac{7}{24}m.$$

Now taking the case where $2 \nmid q$:

$$\begin{aligned} x &= q - \frac{b}{2} + \frac{p}{2} < q + \frac{p}{2} \\ &= \frac{m}{p} + \frac{m}{2q} \\ &= m\left(\frac{1}{p} + \frac{1}{2q}\right). \end{aligned}$$

Now to find the upper bound on x we must maximize p and q using the same reasoning as before. But now q can not be even so the maximizing values are p=5 and q=3 which gives us

$$x \leq m\left(\frac{1}{5} + \frac{1}{6}\right). \text{ Therefore } x \leq \frac{11}{30}m.$$

In both cases they once again violate the properties of m and x and thus we have a contradictions. Therefore m must be a prime of a power of two if b is odd. □

Theorem 1 shows that the only possibilites for m are primes and powers of two. We are now going to show that if b is even then it can also be twice a prime. Taking all of these options, we are now going to show that if $m \geq 2n + b$ then one of these options must be the answer to $\mathcal{S}(n)$.

Theorem 4. *If m is a prime, double prime, or power of two then* $f(x) \not\equiv f(y) \pmod{m}$ *for any* $1 \leq y < x \leq n$

Proof of Theorem 4.

case 1: $m = 2^a$ and b is odd

First assume $1 \leq y < x \leq n$ and note f(x)-f(y) = (x-y)(x+y+b). We can easily attain $1 \leq x - y < n \leq m = 2^a$. Using the fact that b is odd then we know that x-y and x+y+b have opposite parity so one will be odd and one will be even. However we know that $x - y$ is less than 2^a therefore $2^a \nmid x - y$. We also need $x + y + b < 2x + b < 2n + b < n = 2^a$, which leads to $2 \nmid x + y + b$. Thus, 2^a can not divide the product therefore $f(x) \not\equiv f(y) \pmod{2^a}$ therefore n has a distinct remainder.

case 2: $m = p$

Once again assume $1 \leq y < x \leq n$ which can easily become $1 \leq x - y < n \leq m = p$. Taking the inequality we know that p can not divide $x - y$ because it is too small. Therefore p must divide $x + y + b$. However assuming that x and y are not the same and that y is smaller than x we get $x + y + b < 2x + b \leq 2n + b \leq m = p$. Thus p cannot divide x+y+b and therefore $f(x) \not\equiv f(y) \pmod{p}$ therefore n has a distinct remainder.

case 3: $m = 2p$ and b is even

In the final case we again have $x - y < \frac{m-b}{2} < p$ and $x + y + b < 2n + b < m$. Then for m to divide $(x - y)(x + y + b)$, 2 and p must divide the product. However since $x - y$ is smaller than p, it cannot be divisible by p therefore p must divide $x + y + b$. Since b is even we know that either $x - y$ and $x + y + b$ are bith even or odd. However since 2 must divide one they are both even. Since $p \mid x + y + b$ then $x + y + b > p$. Thus $2p|x + y + b$, therefore $m < x + y + b < 2n + b$. However since $2p = m$, p cannot be greater than m thus $p \nmid x + y + b$. Therefore $f(x) \not\equiv f(y) \pmod{2p}$ therefore n has a distinct remainder. □

For the general case the final conclusions that I was able to draw were

$$\mathcal{S}(1,b,n) = \begin{cases} 2^a \text{ or } p & \text{for b odd} \\ 2p \text{ or } p & \text{for b even.} \end{cases}$$

3 Proof for the specific case of $f(t) = t(2t+1)$

Now take the case $f(t) = t(2t+1)$ which says we are looking for $\mathcal{S}$(2,1,n).

Theorem 5. $\mathcal{S}(n) = 2^h$ *is the answer for the minimum* h *such that* $n \leq 2^h$.

Proof of Theorem 5.

Proposition 1:

Note that $f(x) \equiv f(y) \pmod{m}$ if and only if $(x-y)(2(x+y)+1) \equiv 0 \pmod{m}$. Fix h such that $2^h \geq n$. Then f(1),...,f(n) are distinct modulo 2^h and assume $1 \leq y < x \leq n \leq m = 2^h$.

Now taking the previous note into consideration, $f(x) \equiv f(y) \pmod{2^h}$ if and only if $(x-y)(2(x+y)+1) \equiv 0 \pmod{2^h}$ but $2(x+y)+1$ is odd so it is not divisible by any multiple of 2. Thus this is only possible if $x - y \equiv 0 \pmod{2^h}$. However using the previous inequality, $x - y$ is smaller than 2^h, thus $f(x) \not\equiv f(y)$. Therefore n has a distinct remainder. □

Proposition 2: We try to show that m is even assuming $2^{h-1} < n \leq$S$(n) \leq 2^h$

Assume $\mathcal{S}$(n) is even and $m \neq 2q$. By contradiction let $m =$S(n) be odd. Since m is odd, $z = \frac{m-1}{2}$ is an integer.

$$\text{Let } x = \frac{z+1}{2} \text{ and } y = \frac{z-1}{2} \text{ if z is odd.}$$

$$\text{Let } x = \frac{z+2}{2} \text{ and } y = \frac{z-2}{2} \text{ if z is even.}$$

So then $x - y = 1$ or 2 and $2(x+y)+1 = m$. And $f(x) \equiv f(y) \pmod{m}$, therefore $x > n$. But then we have $n < x = \frac{\frac{m-1}{2}+\epsilon}{2}$ for $\epsilon = 1$ or $2 \leq \frac{\frac{m-1}{2}+\frac{m+1}{2}}{2} = \frac{m}{2} < \frac{2^h}{2} = 2^{h-1} < n$ which gives us a contradiction to our original inequlity. Therefore m cannot be odd.

Propostion 3:

Now suppose $m = 2^t p$ for p odd and $t \geq 1$. Assume $2^h - 1 < n \leq m < 2^h$

and $f(1), f(2), ..., f(n)$ are all distinct $\pmod{m}$. We now work towards a contradiction.

For $p \equiv 1 \pmod 4$ and $t = 1$, set $k = 1$. For $p \equiv 1 \pmod 4$ and $t \geq 2$, set $k = 1 = 2^t$. For $p \equiv 3 \pmod 4$, set $k = 3$ if $t = 1$. Otherwise set $k = 3 + p^t$.

Finally, let $x_0 = \frac{kp-1}{4} + 2^{t-1}$ and $y_0 = \frac{kp-1}{4} - 2^{t-1}$. By choice of k, x_0 and y_0 are both integers.

To show $y > 0$ we fix $p = 5$ and $t = 1$.

For $t = 1$ $y = \frac{kp-1}{4} - 2^{1-1} = \frac{kp-1}{4} - 1$.

Then $p \equiv 1 \pmod 4$ meaning that p must be greater than 5 which gives us $\frac{kp-1}{4} - 1 > 0$, continuing on $p \equiv 3 \pmod 4$, which leads to $kp \geq 9$. Finally this shows that $\frac{kp-1}{4} - 1 > 0$.

Now show $f(x) \equiv f(y) \pmod{m}$, where $m = 2^t p$

$x - y = 2^t, x + y = \frac{kp-1}{2}$ this leads to $2(x + y) + 1 = kp$. Therefore $(x - y)(2(x + y) + 1) \equiv 0 \pmod{2^t p}$. Thus we have the same collision and so we know that m cannot be just any even number.

Finally to show $x < m/2$

$$x = \frac{kp-1}{4} - 2^{t-1} = \frac{k\frac{m}{2^t} - 1}{4} + \frac{m}{2p} = \frac{m}{2}\left(\frac{k}{2^{t+1}} + \frac{1}{p}\right) - \frac{1}{4}.$$

So we need

$$\frac{k}{2^{t+1}} + \frac{1}{p} \leq 1.$$

We then fix $t = 1$ and $p = 3$.

When $p \equiv 3 \pmod 4$, we get $\frac{k}{2^{t+1}} + \frac{1}{p} \leq \frac{1}{4} + \frac{1}{5} < 1$.

The next case is when $p \equiv 3 \pmod 4$, which gives us $\frac{k}{2^{t+1}} + \frac{1}{p} = \frac{3}{4} + \frac{1}{p} \leq \frac{3}{4} + \frac{1}{7} < 1$.

However if $t \geq 2$ we still have to satisfy the $\frac{k}{2^{t+1}} + \frac{1}{p} \leq 1$.

When $p \equiv 1 \pmod 4$ it results in,

$$\frac{k}{2^{t+1}} + \frac{1}{p} = \frac{1 + 2^t}{2^{t+1}} + \frac{1}{p} = \frac{1}{2^{t+1}} + \frac{1}{2} + \frac{1}{p} \geq \frac{1}{8} + \frac{1}{2} + \frac{1}{5} = \frac{33}{40} < 1$$

Therefore the condition is satisfied. When $p \equiv 3 \pmod 4$,

$$\frac{k}{2^{t+1}} + \frac{1}{p} = \frac{3 + 2^t}{2^{t+1}} + \frac{1}{p} = \frac{3}{2^{t+1}} + \frac{1}{2} + \frac{1}{p} \geq \frac{1}{8} + \frac{1}{2} + \frac{1}{5} = \frac{33}{40} < 1$$

The final conclusions for this function state that $\mathcal{S}$(2,1,n)$= 2^h$ for h such that $n \leq 2^h$.

4 Proof for the specific case of $f(t) = t(3t+1)$

Taking the case $f(t) = t(3t+1)$ is now saing we are looking for $\mathcal{S}$(3,1,n).

Theorem 6. *$\mathcal{S}(n) = 3^h$ for h min such that $n \leq 3^h$, when $n \geq 9$.*

Proof of Theorem 6.

Let h be such that $3^h \geq n$. Then f(1), f(2),..., f(n) are distinct $\pmod{3^h}$

Suppose $f(x) \equiv f(y) \pmod{3^h} for 1 \leq y < x \leq n < 3^h$ Then (x-y)(3(x+y)+1) is divisible by 3^h. But $3 \nmid 3(x+y)+1$ since it is not a multiple of three, therefore 3 must divide $x-y$. So $x \geq y + 3^h$. Which leads to a contradition since x is already smaller than 3^h

Proposition 1: $\mathcal{S}$(n) must be divisible by 3.

On the contrary, suppose $m =$S(n) and $3 \nmid m$ and that $3^{h-1} < n \leq m < 3^h$, and f(1),...,f(n) are distinct modulo 3^h.

$$\text{Let } z_0 = \begin{cases} \frac{m-1}{3} & \text{when } m \equiv 1(3) \\ \frac{2m-1}{3} & \text{when } m \equiv 2(3). \end{cases}$$

$$\text{Let } x_0 = \frac{z_0 + \epsilon}{2} \text{ and } y_0 = \frac{z_0 - \epsilon}{2}$$

Where ϵ is 1 when z_0 is odd and 2 when it is even.

Then $(x-y) = \epsilon$ and $3(x+y)+1 = 3z_0 + 1 = \begin{cases} m & m \equiv 1 \pmod 3 \\ 2m & m \equiv 2 \pmod 3. \end{cases}$

By definition of m, this implies $x > n$. So $n < x_0 = \frac{z_0+\epsilon}{2} = \frac{lm-1}{3} + \epsilon = \frac{lm}{6} - \frac{1}{6} + \frac{\epsilon}{2} < \frac{lm}{6} \leq m3 < 3^{h-1}$ where $l = 1, 2$ However $3^{h-1} < n$ therefore we have a contradiction, thus $3 \mid m$.

Proposition 2:

So write $m = 3^s p$ for $p > 1$ and $3 \nmid p, s \geq 1$. As always, assume $3^{h-1} < n \leq m < 3^h$ where $h \geq 2$, and $f(1), f(2), ..., f(n)$ are distinct modulo m.

Choose l by

$$l = \begin{cases} 4 + 6k & p \equiv 1 \pmod 6 \\ 2 + 6k & p \equiv 2 \pmod 6 \\ 1 + 6k & p \equiv 4 \pmod 6 \\ 2 + 6k & p \equiv 5 \pmod 6 \end{cases}$$

for k to be determined later. Now let

$$x_0 = \frac{lp - 1 + 3^{s+1}}{6} \text{ and } y_0 = \frac{lp - 1 - 3^{s+1}}{6}.$$

Since $lp = 4 \pmod 6$ and $l \pm 3^{s+1} \equiv 3 \pmod 6$ we see that $lp - 1 \pm 3^{s+1}$ are divisible by 6. Therefore x_0 and y_0 are integers. Then using the knowledge, we can say that$x_0 - y_0 = 3^s$ and $x_0 + y_0 = \frac{lp-1}{3}$. Therefore $3(x_0 + y_0) + 1 = lp \equiv 0 \pmod p$ thus $f(x_0) \equiv f(y_0) \pmod m$.

Now when $y_0 > 0$

$$y_0 = \frac{lp - 1 - 3^{s+1}}{6},$$

so $y_0 > 0$ iff $lp - 1 > 3^{s+1}$, where l is to be chosen later. Therefore we get $lp > 3^{s+1} + 1$

We need $x_0 < m/3$

$$= \frac{lp - 1 + 3^{s+1}}{6} < \frac{m}{3}$$

This is only true iff $lp - 1 + 3^{s+1} < 2m$ which can be written as,

$$\frac{lm}{3^s} - 1 + \frac{3m}{p} < 2m \text{ which simplifies to } m\left(\frac{l}{3^s} + \frac{3}{p} - 2\right) < 1.$$

We will now choose l such that $\frac{l}{3^s} + \frac{3}{p} - 2 \leq 0$, that is $lp \leq 3^s(2p - 3)$. Now, if lp is so chose then $f(x) \equiv f(y)$, then $x_0 > n$. But $x_0 < m/3 < 3^h/3 = 3^{h-1} < n$. Therefore we have a contradiction. Our proof is complete once the conditions on lp are met. We need $lp > 3^{s+1} + 1$ and $lp \leq 3^s(2p - 3)$.

We need $lp > 3^{s+1} + 1$ and $lp \leq 3^s(2p - 3)$. That is $l > \frac{3^{s+1}+1}{p}$ and $l \leq 3^s(2 - \frac{3}{p})$.

Suffice (to pick $l = \epsilon + 6k$) to have $\frac{3^{s+1}}{p} + 6 < 3^s(2 - \frac{3}{p})$. For then we can pick k to have $\frac{3^{s+1}+1}{p} < l = \epsilon + 6k < 3^s(2 - \frac{3}{p})$. That is $3^{s+1} + 6p < 3^s(2p - 3)$, which is $2 * 3 + 1 < (2 * 3^s - 6)p$. And from this point just working out the algebra, we obtain $\frac{2*3^{s+1}+1}{2*3^s-6} < p$. Then we can now begin to break it all down and get

$$\frac{2 * 3 * 3^s - 18 + 18 + 1}{2 * 3^s - 6} = 3 + \frac{19}{2 * 3 - 6} < p$$

Now if $s \geq 2$, the previous statement is $\leq 3 + 19/12 < 5$. Therefore everything until now worked for $p \geq 5$ and $s \geq 1$.

Now we do the case where $s = 1$ and $m = 3p$ where $3 \nmid p$.

We assume $e^{h-1} < n \leq m < 3^h$ and that $f(1), f(2), ..., f(n)$ are distinct module m. We can check the small cases by hand, and so may assume $p > 10$ and $m > 24$. Choose $r = 1, 2$ or 4 so that $rp \equiv 4 \pmod 6$. (So if

$p \equiv 1$ then $r = 4$, if $p \equiv 2$ then $r = 2$ etc.) Set $x = \frac{rp+8}{6}$ and $y = rp - 106$. By the choice of r, x and y are both integers, and because $p > 10$, both are positive integers.

Now $x - y = 3$ and $3(x + y) + 1 = rp$. Thus $f(x) \equiv f(y) \pmod{m}$, and so $n < x$. We have $x = \frac{rp+8}{6} = \frac{rm}{18} + \frac{8}{6} \leq \frac{4m}{18} + \frac{8}{6}$. And because $m > 24$ is it easy to see that this last value is greater than $m/3$. But then $3^{h-2} < n < x < m/3 < 3^{h-1}/3 = 3^{h-1}$. Therefore we have a contradiction. From which we can now say that m is not just a multiple of 3. $\square$

The final results of the last case of $\mathcal{S}$ (3,1,n)= 3^h for h such that $n \leq 3^h$.

5 References

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6 Acknowledgements

Jim Sauerberg
Saint Mary's College of California
Summer Research Program 2012