Saint Mary's College of California

Department of Mathematics

Senior Thesis

Ramsey Theory: Complete Disorder?

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Math 193 – Senior Seminar **Senior Essay**

Prof. Kathy Porter **Due: May 7, 2017**

Introduction:

One of my goals for my senior essay was to get at least 367 people to read my thoughts on Ramsey Theory. If this occurred, I could guarantee that at least two of my readers shared the same birthday. That seems like a crazy idea, but it will be addressed in my paper along with so much more. Do you ever think to yourself, "What a coincidence," when two seemingly seperate things end up having some commonality or structure? The mathematician Frank Ramsey would not say "what a coincidence" but instead would say "what a big enough set." Ramsey would be referencing that it may not have been a coincidence at all because you might have allowed the initial set to be large enough such that commonality between the subsets was guaranteed. Mathematically, this is saying that if you choose an initial set with enough elements, you are guaranteeing the subsets will have structure between them. Logically this makes sense because if you were looking for guaranteed structure within a set, one would probably start looking at a set with many elements rather than one with few elements. This is the basic concept of Ramsey Theory, as it is a branch of mathematics that studies how order must appear in certain conditions and how complete disorder is nearly impossible. We can't quite say that complete order is always guaranteed between sets, but we can say that complete disorder is very unlikely. Within Ramsey Theory, both Ramsey's and Van der Waerden's Theorems are incredibly useful and applicable.

Basically, Ramsey's Theory is "the study of the preservation of properties under set partitions" [13] pg.1. Another way to say this would be that if we say that a particular set S has a property P, then it is true that if S is partitioned into finitely many subsets, one of the subsets must also have property P [13] pg. 2. I am going to give a brief background on both Ramsey and Van der Waerden, introduce some important definitions around Ramsey Theory, then dive in to breaking down both of their theorems, before concluding with some examples and applications within Ramsey Theory. The central theme of this paper is to understand and find this order and structure between different sets that Ramsey said was nearly inevitable.

Biographies/Background:

The two main mathematicians I will be focusing on in this paper are Frank Ramsey and Bartel van der Waerden. [6] Frank Ramsey was born in Cambridge, England in 1903, and he died at the young age of 26 in 1930. He completed his secondary school education in 1920 at Winchester. In 1926, he was appointed as a university lecturer in mathematics at King's College and later became a Director of Studies in mathematics there. According to his students, he was known for his enthusiastic and clear lectures on the foundations of mathematics. His first major published work was "The Foundations of Mathematics" in 1925. In this work, Ramsey accepted the claim by Bertrand Russell and Alfred Whitehead from "Principia Mathematica," that mathematics is a part of logic. Ramsey always made it his goal to show logic's place within the world of mathematics. His second paper was called "On a problem of formal logic" which was read to the London Mathematical Society in 1928, but wasn't published until 1930. His paper examines methods for determining a logical formula's consistency and it includes some combinatorics theorems that have led to the study of a whole new area of math called Ramsey theory. This celebrated paper stimulated the study of graph theory, and Ramsey's Theory was became an established and growing branch of combinatorics. The picture below on the left is Frank Ramsey in his 20's before his life was cut short. [6] [8]

Frank Ramsey

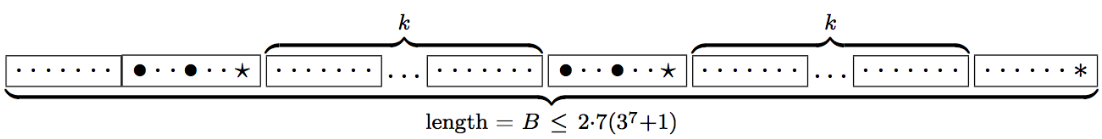
Bartel Van der Waerden

[7] Bartel van der Waerden, pictured above and on the right, was born on February 2 in Amsterdam, Netherlands and also in the year 1903. Interestingly enough, growing up Bartel was told to not read his father's mathematical books because his father wanted him to go and outside and play outside like a a normal child. However, this rule only intrigued Bartel more to discover the world of math. He studied mathematics at the University of Amsterdam in 1919 at just the ripe age of 16. He studied topology and invariant theory in his early university studies. He wasn't always a pleasure to have in class because math came so easy to him, and he made his presence known through conceited and disruptive remarks. Throughout his career, Bartel worked on such topics as algebraic geometry, abstract algebra, combinatorics, quantum me-

chanics, and the history of mathematics. [4] It is worth noting that the Van der Waerdens Theorem was proved in 1927, a year earlier than Ramseys, yet Van der Waerden's central theorem is still under the over-arching topic of Ramsey Theory.

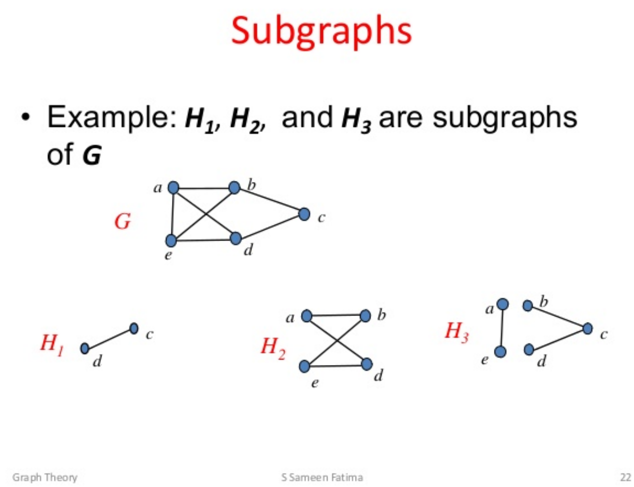
Note: The following definitions are all relevant to Ramsey's Theory, but also important for my paper itself. Some of these definitions are necessary for a baseline understanding of Ramsey Theory. The more specific definitions will occur later in my paper immediately before they are being used or applied. Read through these foundational concepts now, and make sure they make sense before moving on to the next section. This is a good section to refer back to because it will clear up any confusion later on in my paper about any terms or concepts.

Definition 1: [15] *Combinatorics* is the branch of mathematics studying the enumeration, combination, and permutation of sets of elements and the mathematical relations that characterize their properties. I define this because I mentioned it in the introduction/background, and because it is an important part of Ramsey's theory. I feel that a basic understanding of combinatorics is necessary to truly understand Ramsey Theory. Within combinatorics, there is an important concept called *coloring*, which is crucial when it comes to proving Ramsey or Van der Waerden's Theorem. Coloring refers to how we assign each element in a set or group a different color; they could all be the same color or we could have a nice mix of all the colors. It is worth noting that the number of ways to color n objects with r colors is r^n, and this becomes very important in the Van der Waerden section. For those confused, the image below is a breakdown of a block being colored with either a dot, big dark circle, or a star. This image is also relevant because it shows that when you color certain elements within a block, they don't actually have to refer to colors because this elements are being "colored" with three different shapes. Refer back to this figure if you get lost along the way of any Van der Waerden coloring proofs or if you get confused at what it exactly means for a block of elements to be colored. [10]



Note: These next four definitions are at the ground level for understanding graph theory and how it relates to Ramsey's Theorem and are found on [13] pg 7. They all work together and are necessary to prove Ramsey numbers or bounds on certain Ramsey parameters.

Definition 2: A *graph* $G = (V, E)$ is a set V of points, called *vertices*, and a set of E of pairs of vertices, called *edges*. Also, a *subgraph* $G' = (V', E')$ of a graph $G = (V, E)$ is a graph such that $V' \subset V$ and $E' \subset E$. All of the subgraphs must be contained within the graph and the subgraph cannot have more points or vertices than the original graph. Below are some examples of subgraphs. [1]



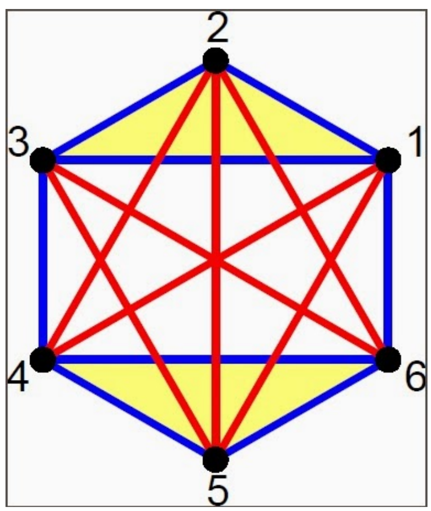
A *complete graph on n vertices*, denoted K_n, is a graph on n vertices, where every pair of vertices is connected by an edge. Within subgraphs, there is something called an *edge-coloring* of a graph, which is an assignment of a color to each edge of the graph. If a graph's edges are all colored and they are all the same color, then it is called a *monochromatic graph*. This definition will come in handy when we use graph theory and it is necessary to prove a Ramsey theorem problem.

Breakdown of Ramsey Theory

When approaching a Ramsey Theory problem, we know that our initial object of interest needs to appear as a subset of some larger set. In addition, we also need to determine how large our superstructure must be so that no matter how we partition it into a given number of parts, one of the partitions (or subsets) still has that desired substructure. All of the theorems under Ramsey Theory are all just different ways of saying the same thing, which is if you make something big enough, you have a better chance at finding structure and organization within it when you break it up. Ramsey Theory also deals with the lower and upper bounds for the initial large group in order to have certain subgroups of it. I want to focus on the first two theorems of Ramsey Theory, which are Ramsey and Van der Waerden. [14] pgs. 9-10: Graham wrote about the six important Ramsey-type theorems, but here are the two that I will be focusing on in my essay:

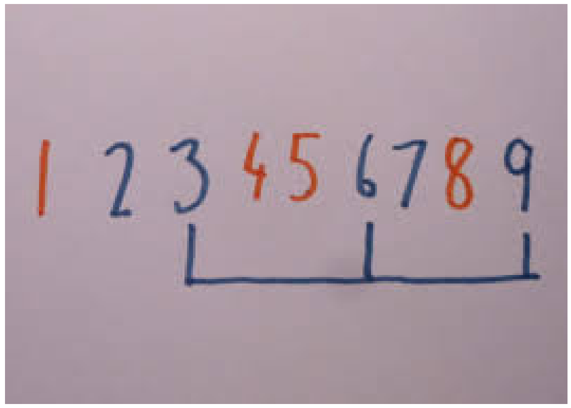
1) **Ramsey's Theorem**: *For all l, r, k there exists a graph n_0, so that for any $n \geq n_0$, if the edges of $[n]^k$ are r-colored, then there exists a monochromatic $[l]^k$ subgraph.*

This is saying that no matter how many colors you use or how big you want your subgraphs to be, there is some main graph that it will definitely be contained within. The image below reveals that 123 and 456 represent monochromatic subgraphs on 3 vertices because these vertices all have blue edges connecting them. This theorem works no matter the number of colors that are used, but I think Ramsey's theorem for two colors is itself a satisfying theorem. I like the theorem for two colors even if it restricts our ability to expand the number of colors, and that theorem is presented later. [12]



2)**Van der Waerden Theorem**: *For all k, r, there exists n_0 so that for $n \geq n_0$ if $[n]$ is r-colored there exists a monochromatic arithmetic progression $\{a, a+d, ..., a+(l-1)d\} \subset [n]$ of length k.*

Another way of saying this theorem is that for all positive integers k and r, there exists an integer $W(k, r)$ so that if the set of integers $\{1, 2, ..., W(k, r)\}$ is partitioned into r classes, then at least one class contains a k-term arithmetic progression. Basically, no matter the paramters k and r that are chosen, Van der Waerden says that there is definitely an integer that exists for these parameters. Below is a very simple example of the integers from 1 to 9 written out and colored either red or blue. You can see the blue 3-term arithmetic progression. This isn't enough to prove that the integer 9 can always be broken down in this way, but it is an example of how Van der Waerden's Theorem can be visualized. [5]

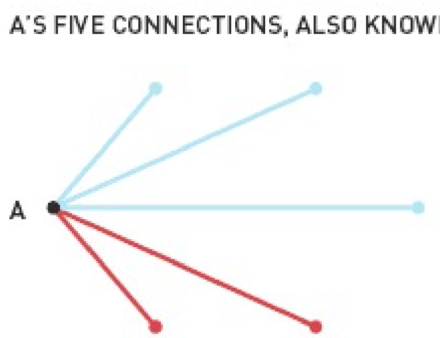


Ramsey's Theorem:

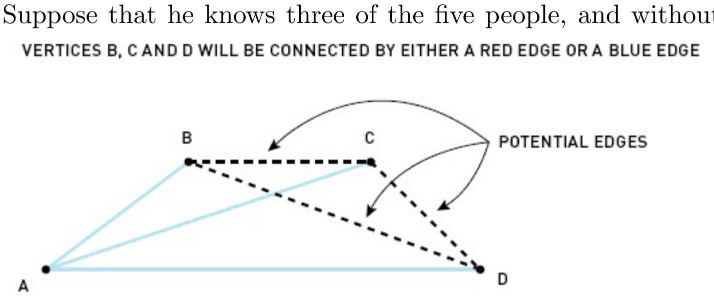
Statement: [13] pg.3 *The Pigeonhole Principle* states that if a set of n elements is partitioned into r disjoint subsets where $n > r$, then at least one of the subsets contains more than one element. This is an important result because everything within Ramsey theory requires some sort of understanding of the Pigeonhole Principle. It is amazing the number of things that revolve around this principle because even the simple game of Musical Chairs involves this concept. It relates to the Pigeonhole Principle because there are n players running around a circle with only $n - 1$ chairs once a chair is removed during the round. So, unless a player sits on someone else's lap, a player will be chairless, and thus the competitive game of Musical Chairs relates to the Pigeonhole Principle.

Ramsey's Theorem is at times considered a refinement of the Pigeonhole Principle. This is because while the Pigeonhole Principle guarantees certain elements in a pigeonhole, Ramsey's Theorem also guarantees a certain relationship between these elements. At the heart of understanding Ramsey's Theorem is the simple and relatable Party Problem. The Party Problem may be considered the first nontrivial example of Ramsey Theory. The problem deals with determining how many people need to be at your party in order to have a certain numbered subgroup of mutual friends or mutual strangers.

Specifically within a six person party, the Party Problem claims there must be either at least three people who all know each other, or at least three people who are all complete strangers. We say that a blue line between people represents that they have met before while a red line represents two strangers at the party. If we relate each person at the party to a specific vertex, this then becomes a graph theory problem with six vertices and we want to find a monochromatic subgraph of either a red or blue triangle. Due to the Generalized Pigeonhole Principle, we know Person A must either know or not know three of the other five people.[11]



Suppose that he knows three of the five people, and without loss of generality, let us say he knows B, C, and D. [11]



If some pair of these three knows each other, such as C and D, then the blue triangle ACD will represent three people who mutually know each other and we are done. On the other hand, if no pair of the three know each other, then those are the three strangers we are looking for and BCD is our red triangle. Either way we are done because we have found our monochromatic subgroup of three. The argument is the same if A begins with not knowing three of the five people. Once you understand the Party Problem, you find yourself looking at your first Ramsey number which is that $R(3,3) = 6$, but more on that later.

Ramsey Theorem is all about expanding this simple problem to figure out how many people you would need to guarantee a group of four strangers, five mutual friends, or something beyond that. The possibilities are endless with this theorem, but it does get rather complicated beyond single digit subgroups or if you want to introduce a third subgroup, such as acquaintances. For now, I am going to keep it simple and restrict the number of colored subgroups to just two, which gives us our first important theorem. [13] Landman and Robertson on pg. 8 outlined a very clear and concise way to prove Ramsey's theorem simply by setting the number of colors used to two.

Theorem 1 Ramsey's Theorem for Two Colors: *Let $k, l \geq 2$. Then there exists a least positive integer $R = R(k,l)$ such that every edge-coloring of K_R, with two colors such as red and blue, produces either a red K_k subgraph or a blue K_l subgraph.* [13]

Proof. :

[13] pg. 8 Let $k, l \geq 2$. We want to show that there exists a least positive integer $R = R(k,l)$ such that every edge-coloring of the larger graph K_R, with two colors such as red and blue, produces either a red K_k subgraph or a blue K_l subgraph. We want to show that no matter what the parameters k, l are, there is an integer that will definitely exist and we will prove this by induction. First, take note that $R(k,2) = k$ for all $k \geq 2$ and $R(2,l) = l$ for all $l \geq 2$. This is because if we have a complete graph with k vertices, it will be either be colored all one color, and our complete subgraph would have k vertices, or at least one edge is the other color, in which case we would have a complete subgraph with at least 2 vertices. That tells us that $R(k,2) \geq k$, and we know $R(k,2) \leq k$ because the subgraph cannot have more vertices than the larger graph itself; thus, we have shown that $R(k,2) = k$ and $R(2,l) = l$ can be proved the same way. We proceed to prove that there always exists a least positive integer $R = R(k,l)$ via induction on the sum $k+l$, having taken care of the case when $k+l \leq 5$ or if either k, l equal 2. Hence, we let $k+l \geq 6$, with $k, l \geq 3$. Since we are using induction, let us next assume that $R(k,l-1)$ and $R(k-1,l)$ exist; we claim that $R(k,l) \leq R(k-1,l) + R(k,l-1)$. So, we set $n = R(k-1,l) + R(k,l-1)$ and we consider the graph K_n. Now, we select one particular vertex, v from this graph K_n. Then, there are $n-1$ edges coming from v to the other vertices. Since we set blue and red to be the two colors in the assumption, let A be the number of red edges and B be the number of blue edges coming out of v. We know either $A \geq R(k,l-1)$ or $B \geq R(k,l-1)$ because if $A < R(k-1,l)$ and $B < R(k,l-1)$, then $A+B \leq n-2$, which contradicts when we said there were $A+B = n-1$ lines coming out v. Without loss of generality, let us assume $A \geq R(k,l-1)$. Let V be the set of vertices connected to v by a red edge, so that $|V| \geq R(k-1,l)$. K_V either contains a red K_{k-1} subgraph

or a blue K_l subgraph due to the previous inequality and inductive hypothesis. We are done if it contains a blue K_l subgraph. However, let us say it contains a red K_{k-1} subgraph; then by connecting v to each vertex of this red subgraph we have a K_k subgraph because v is connected to V by only the red edges. So, we have proven it will either be a K_k or K_l subgraph which is what we wanted to show because we needed to make sure that at least one of those subgraphs will always occur. To recap our induction steps, we showed that the statement was true at $k, l = 2$, then we showed that whenever it was true at $k-1, l-1$, it was also true at k, l as well; Since we showed these steps, we know the statement will be true for all k, l, or no matter what parameters of subgroups are chosen. Thus, our proof is complete. □

[14] pg.92 From the inequality $R(k,l) \leq R(k-1,l) + R(k,l-1)$ in the previous proof, Graham and Rothschild determined that you can say that $R(k,l) \leq \binom{k+l-2}{k-1}$. You will find out later that $R(3,4) = 9$, so let us test this out. So, $R(3,4) \leq \binom{3+4-2}{2}$ and $\binom{5}{2}$ is actually 10 which is obviously more than 9. That bound is higher than that lowest upper bound, but the important thing is that there is a way to determine an integer for a bound that definitely works for specific parameters. We will show later that for diagonal Ramsey numbers, which will be defined later, this formula would have given the exact Ramsey number. Ramsey's Theorem is about the existence of a number no matter the parameters k, l of the subgroups, and now we move on to the Ramsey numbers that are known today.

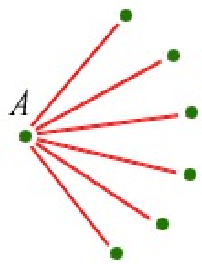
The numbers $R(k,l)$ are referred to as 2-color Ramsey numbers because k, l represent the two parameters. As I said before, once we figured out the Party Problem, then we knew $R(3,3) = 6$ is an example of one of the elusive Ramsey numbers. Until 1930, there were no other Ramsey numbers that had been officially discovered. [13] pg. 9 Currently, the total list of known Ramsey numbers includes $R(3,3) = 6, R(3,4) = 9, R(3,5) = 14, R(3,6) = 18, R(3,7) = 23, R(3,8) = 28, R(3,9) = 36, R(4,4) = 18, R(4,5) = 25, R(3,3,3) = 17$. It is worth noting that $R(3,3,3) = 17$ is the only nontrivial known Ramsey number involving more than 2 colors. It is important to realize that the order of the parameters k, l do not matter because $R(3,5) = R(5,3) = 14$. This is because in both cases you are looking for either a monochromatic subgraph on 3 or 5 vertices. So, to breakdown $R(4,5) = 25$, you would need a twenty-five person party to guarantee there is either a group of at least four mutual acquaintances or at least five mutual strangers. The fact that is equals 25 is significant as well because it means that there was at least one counter-example where 24 vertices for the large graph was not enough, and any graph with more than 25 points works as well. Next, instead of simply listing the known Ramsey numbers, I will take it a step further and prove that $R(3,4) \leq 10$ because 10 is a possible bound for these parameters. Previously, I have showed this through the choose function, but now I will give an alternative proof using Graph Theory. [9]

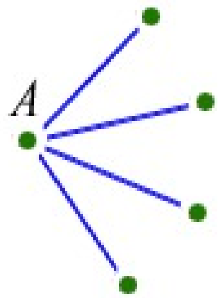
Proof 1: *If we let the parameters of the subgraphs be* 3 *and* 4*, then a graph of* 10 *points is enough to guarantee the existence of at least one of these monochromatic subgraphs. In other words,* $R(3,4) \leq 10$.

Proof. :

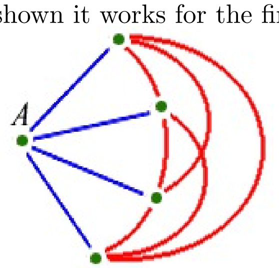
We want to show that $R(3,4) \leq 10$ by showing that 10 is an upper bound that does work and then $R(3,4) > 8$. First, I want to show that 10 points is enough. Let us say red represents the subgroup of 4 strangers and blue represents the subgroup of 3 friends. If we have 10 points, pick a particular point and note that it will have 9 lines coming out of it.

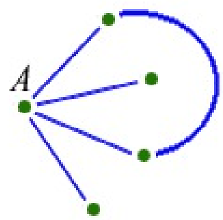
There must be either at least 4 blue ones or at least 6 red lines and let's look at these two cases.



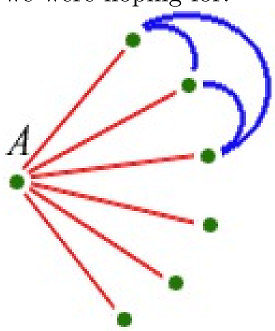


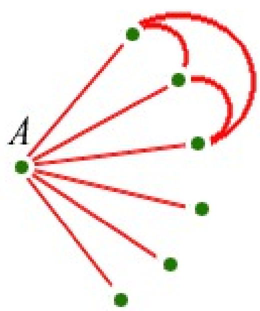
In the first case of 4 blue lines coming out the vertex A, if one of the outer lines in the figure is blue, then we have a blue triangle of 3 friends. If none of the other lines are blue, then we have our group of 4 red strangers. Either way, we have shown it works for the first case of 4 blue lines.



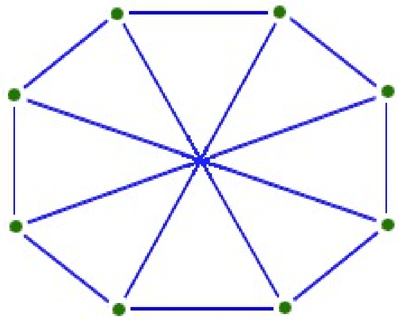


Finally, we look at case 2, which is 6 red lines emerging from point A. We have already discovered that $R(3,3) = 6$ from the Party Problem, so out of these 6 points, 3 must form either a red or blue triangle. Either way, we have shown that it works for the second case of 6 red lines. In both of these two cases, we have our 3 friends or 4 strangers, which is what we were hoping for.





So, now we have set our upper bound for $R(3,4)$ at 10, but we must check to see if the bound can be lowered to 8 or 9. On the other hand, we can see that 8 points would not be enough. If you look at the following diagram for this specific counter-example, all the blue lines are shown and the remaining lines are assumed to be red.



No 3 points form a blue triangle, and no 4 points form a red subgraph. Thus, since we have bounded $R(3,4)$ to be more than 10, and we know 8 does not work, we have shown that $R(3,4) \leq 10$. Specifically, since 8 points is definitely not enough, we have shown that $9 \leq R(3,4) \leq 10$. □

Note: There are a few rules that come along once you determine a bound for certain Ramsey parameters. [14] You could exchange the places of red and blue, so the same argument could be made that $R(4,3) \leq 10$ as well. This also means that 10 works for $R(3,3)$ because if four people mutually know each other, you can easily just delete one of them. Furthermore, since 10 works for $R(4,3)$, then we know 11 will work as well because you could just ignore the eleventh person.

Elusiveness of Ramsey numbers:

One might think if you sit down and compute for long enough, a person can stumble upon some of the elusive Ramsey numbers, but these Ramsey numbers are difficult to compute. Although we don't know the exact Ramsey number for certain parameters, it is quite amazing that Ramsey theory can give us upper bounds for $R(m,n)$ for any natural numbers m and n, similar to how we bounded $R(3,4)$ at 10. [9] A perfect example of this elusiveness is that our present knowledge tells us that $R(5,5)$ is somewhere between 42 and 54. When we say the Ramsey number is somewhere between 42 and 54, we are saying that we know 41 points is not enough to find subgroups on five vertices because there is at least one counter example, but we know 54 is definitely enough points to prove this. Some might be surprised that we can't be more accurate about $R(5,5)$ with the innovations we have had. However, let us say we wanted to show that $R(5,5)$ is bounded at 53, which is just below our upper bound of 54. In other words, we would have to show that 53 points will always guarantee the existence of 5 points all connected together by lines of one color. To do this, we would have to look at all the possible ways of coloring lines either red or blue between 53 points. Do you know how many ways there are? First we need to know the number of lines, which, by a well-known choose formula, is $(53 \times 52)/2 = 1378$. Each line can be colored either red or blue, so the number of possible colorings is 2^{1378}, or over 10^{400}. Just a fun tidbit of information to put this into perspective... this number is much larger than the number of particles in the known Universe, which is about 10^{80}. Due to this computational complexity, we definitely can not determine $R(5,5)$ equals exactly 53 in our present age of mathematical technology. It even seems unlikely in the near future but my fingers are crossed. Technological advancements might be the biggest benefit for the field of Ramsey theory.

Graph Ramsey Theory:

Next, we move on to something that is central to every Ramsey number proof, and is basically a rephrasing of Ramsey Theory itself. We have been doing graph Ramsey Theory without realizing it, but the idea behind graph Ramsey theory is basically as follows: [14] pg.138 For an arbitrary and fixed graph G, we would like to determine the fewest edges a graph could have, or the smallest integer $r = r(G)$ so that, no matter how you 2-color K_r, a monochromatic subgraph isomorphic to G is always formed. [14] pg.115 Similarly, there is also the edge-induced graph theorem which states that for all $G, r \geq 0$, there exists a initial larger graph H where if the edges of H are all r-colored, then as a result there exists a produced subgraph G with monochromatic edges, or edges all the same color. This has to do with finding a monochromatic subgraph as apart of a larger graph, which is just how we prove a specific Ramsey number but within graph theory. Interestingly enough, proofs of Ramsey numbers are direct problems of graph theory.

Note: $R_2(k) = R(k, k)$ is the least positive integer such that every 2-coloring of the edges of the complete graph on $R(k, k)$ vertices admits a monochromatic complete graph on k vertices. In other words, $R_2(3) = R(3, 3) = 6$, as we have shown before in the Party Problem. This is also another way to phrase what is called a diagonal Ramsey number because it involves finding a monochromatic subgroup using the one parameter. Here is the definition for these diagonal Ramsey numbers.

Definition 3: [13] pg.280 The numbers $R(k, k)$ are called *diagonal* Ramsey numbers, and when $k \neq l$, $R(k, l)$ is called an off-*diagonal* Ramsey number. So, when the parameters k, l are equal, then the result would be a diagonal Ramsey number, such as $R(3, 3)$, but $R(3, 4)$ would be considered an off-diagonal Ramsey number. This is just a piece of trivia that isn't crucial to my paper, but it is interesting to know that there are different categories of Ramsey numbers. Also, here is a useful application that helps determining diagonal Ramsey numbers only.

Dealing with the definitions of diagonal and non-diagonal Ramsey numbers, it is worth noting that it is much easier to find the exact number of diagonal Ramsey numbers than for non-diagonal Ramsey numbers. The reason $R(3, 4)$ didn't equal exactly 9 from the last time we used this choose function is because it is not a diagonal Ramsey number, as its parameters don't equal each other. [3] It is easier because to find the exact lowest upper bound for a diagonal Ramsey number, there is a simple function that only needs the one parameter and it was written about by Martin Gould. The equation for diagonal Ramsey numbers becomes

$$R_2(s) = \binom{s + s - 2}{s - 1} = \binom{2(s - 1)}{s - 1}.$$

To illustrate this we look at $R_2(3)$ which we know to be 6. So, using the equation, we see that

$$R_2(3) = \binom{4}{2}$$

$=(4 * 3)/2 = 6$. Diagonal Ramsey numbers are easier to work with because you only need one parameter to determine the exact integer that allows it to occur. This formula will work for all diagonal Ramsey numbers, as it requires just the one parameter for an input.

For Van der Waerden:

Definition 4: [13] pg.12 A sequence of the form $\{a, a+d, a+2d, ..., a+(k-1)d\}$, where $a \in \mathbb{Z}$ and $d \in \mathbb{Z}^+$ is a k-term arithmetic progression. Finding pure arithmetic progressions is something you have been seeing since first grade, when you were asked to pick the next number in the sequence of $3, 5, 7, ...$ Instinctively, we know the next number to be 9 but that is because of the concept of arithmetic progressions. This definition and concept is crucial to recognize when it comes to proofs of Van der Waerden numbers.

Monochromatic arithmetic progression is another aspect of Ramsey's theory. In other words, if all positive integers are partitioned into two classes, similar to the two colors in Ramsey's theorem, then at at one of those classes will contain arithmetic progressions of a given length. Within an arithmetically progressing set, all the set's elements will be equi-distant from each other.

Theorem 2 Van der Waerden's Theorem: [13] pg. 25 *Let* $k, r \geq 2$ *be integers. There exists a least positive integer* $w = w(k, r)$ *such that for any* $n \geq w$*, every* $r-$ *coloring of* $[1, n]$ *admits a monochromatic arithmetic progression of length* k.

In other words, you break up the integer n into r classes and you should be able to find a k term arithmetic progression. [13] pg.29 Another way to think about Van der Waerden's Theorem directly related to arithmetic progressions is if we let $r \geq 2$, no matter how we r-color $\mathbb{Z}^+$ and given a subset $S = \{s_1, s_2, ..., s_n\} \subset \mathbb{Z}^+$, there definitely exists integers a, d such that $a + dS = \{a + s_1 d, a + s_2 d, ..., a + s_n d\}$ is monochromatic. It is clear that the sequence $a + dS$ is the arithmetic progression that you are searching for in Van der Waerden's Theorem.

The great thing about Van der Waerden's theorem is that similar to Ramsey's, no matter the parameters that are chosen, there is some existing integer that allows these parameters to occur. In other words, no matter how many colors are chosen (r) or how long of an arithmetic progression we are looking for (k), there is some existing number that allows this to occur. So, there is an upper bound that definitely works for certain parameters, but it is complicated to find the exact lowest upper bound.

[13] pg. 31 Just like Ramsey numbers, there aren't very many known van der Waerden numbers either. The nontrivial numbers of $w(k, r)$ is $w(3, 2) = 9, w(4, 2) = 35, w(5, 2) = 178, w(6, 2) = 1132, w(3, 3) = 27, w(4, 3) = 293$, and $w(3, 4) = 76$. [14] pg. 30 After this paragraph is a scholarly breakdown from the book by Graham, Rothschild, and Spencer that proves that 325 is a large enough integer to break up into two classes and find an arithmetic progression of length three. The number 325 was not randomly chosen; instead, it was a specific choice to choose this integer as you will see in this proof. They wanted to break up the integer into blocks of length 5, and 325 is divisible by 5. Furthermore, since there are only $2^5 = 32$ ways to 2 color a box, they needed at least the integer $33 * 5 = 165$ to guarantee that 2 blocks were colored the same way by the Pigeonhole Principle. However, our explanation is based on guaranteeing a 3-term arithmetic progression. So, we choose a third box that is equally distant from the second box as the second is from the first box. This is because the boxes themselves create the foundation for the arithmetic progression and then we simply

make conjectures about the last element. And no matter if the last element is red or blue, we would have still found an arithmetic progression of length 3. Worst case scenario, this last block we are looking at it would have to be in the next 32 blocks because it is simply equidistant from the second box, which is how they stumbled upon the integer $325 = 65 * 5$. So, we want to prove that the show that if you are given $W(3,2)$ as parameters, 325 definitely works as bound. [14]

Proof 2: *The integer* 325 *definitely works as a bound for* $W(3,2)$.

Proof. :

We want to show that the integer 325 works for $W(3,2)$. First, we break the integer into two classes and remember that we are looking for a 3-term arithmetic progression. Note that 5 times 65 equals 325. So, Divide 325 into 65 blocks of length five such that $[1,325] = [1,5] \bigcup [6,10] \bigcup ... \bigcup [321,325]$; Since they are split into two classes, it means they are 2-colored; there are $2^5 = 32$ possible ways to 2-color a block B_i which is simple combinatorics; thus, of the first 33 blocks, some pair of blocks must be 2-colored in exactly the same way due to the pigeonhole principle. Let us arbitrarily say that blocks B_{11} and B_{26} are two colored the same way and let us say the two colors are red and blue. We know that of the five elements in each block, at least three are red or at least three are blue. In $B_{11} = \{51, 52, 53, 54, 55\}$, if we look at the first three elements $(51, 52, 53)$, at least two must have the same color again because of the pigeonhole principle, say j and $j+d$. Because of this, we also know $j+2d$ also belongs to B_{11}. If $j+2d$ has the same color as j and $j+d$, then we are done because $j, j+d, j+2d$ would represent an arithmetic progression of length 3. Thus, let us assume that $j+2d$ has the other color. We could do the same thing for all the other ways to 2-color the first three elements of the box, but let us arbitrarily say that B_{11} and B_{26} each have the coloring of $\{RBRRB\}$ since we said that $j+2d$ will be the other color. It is important to note that the numbers in B_{26} are $\{126, 127, 128, 129, 130\}$ and remember that we said it would be two-colored the same as B_{11}. Next, we need to arithmetically progress 15 more blocks from B_{26} because it is 15 blocks away from B_{11}. Now, we look at the coloring of B_{41}. Actually, because the three blocks themselves are arithmetically progressing, the first four elements of B_{41} don't matter for this proof so it has the box coloring of $\{....R \text{ or } B\}$. We see that if the final integer $205 \in B_{41}$ is blue, then $55, 130, 205$ is a blue arithmetic progression as the last elements in all the boxes are the same color. If it is not blue and is red instead, then $51, 128, 205$ is a red arithmetic progression as the first element in B_{11}, the third element in B_{26}, and the fifth and last element in B_{41} are all red. Either way we are done because we have found our 3 term arithmetic progression no matter what. We have proven that no matter how you want to 2-color the integers from $[1,325]$, there will an arithmetic progression of length 3. Ergo, we have shown that 325 works as a bound for $W(3,2)$. □

It is interesting that this proof bounds at $W(3,2)$ at 325, but the actual number is 9. It shows just how hard it is to prove the exact Van der Waerden number. I had hoped to do my own proof for a Van der Waerden bound, but I feel my time was better well spent explaining just how difficult a Van der Waerden proof is. Even if we just increased either of the parameters by one from the previous proof, the proofs for bounds of $W(3,3)$ and $W(4,2)$ are incredibly more complicated. To find a 3 term arithmetic progression involving 3 colors, we would either have to say there are 3^5 or 243 different ways to 3-color a block. Also, if we used the same format as the previous bound proof, we would let each block be of length 7 and then we would have to make some unsound assumptions about the color ordering for the final proof. On the other hand, if we to try to find a 4 term arithmetic progression involving 2 different colors, we would need to extend our blocks to 13 elements to look for a 4 term arithmetic progression since we know only 9 elements are needed for a 3 term arithmetic progression. And then we are looking at 2^{13} possible ways of 2 coloring a 13 element box. Therefore, it is clear how hard it is to prove that any integer is a bound for specific paramters, let alone the exact Van der Waerden number itself. This is a table of known Van der Waerden numbers and bounds and you can see how much these Van der Waerden numbers greatly increase as the paramters barely increase. [2]

$r\backslash k$	**3**	**4**	**5**	**6**	**7**	**8**	**9**
2	9 [3]	35 [3]	178 [14]	> 1131 [9]	> 3703 [11]	> 7484 [11]	> 27113 [11]
3	27 [3]	> 292 [11]	> 965 [11][2]	> 8886 [11]	> 43855 [11]	> 238400 [11][3]	
4	76 [1]	> 1048 [11]	> 10437 [11]	> 90306 [11]	> 387967 [11][3]		
5	> 125 [4]	> 2254 [11]	> 24045 [11]	> 246956 [11][3]			
6	> 207 [11]	> 9778 [11]	> 56693 [11][3]	> 600486 [11][3]			

[2]Landman and Robertson [10] refer to an untraceable lower bound $W(3,5) > 1209$.
[3]Unpublished lower bounds which could be established using the method presented in [11].

Although Van der Waerden numbers are hard to prove exactly, there may be a pattern with particular Van der Waerden parameters. So for, $w(k, r)$, it is worth noting that $w(3, 2) = 9$ and $w(3, 3) = 27$; these numbers both happen to be powers of 3 which is their parameter for k. Most people would then venture to say that $w(3, 4)$ would be around the ballpark of the number 81. It is actually 76, which is barely less than 3^4, but it is in the same vicinity. One might conjecture that when looking for a 3 term arithmetic progression, you could put some bound on what the Van der Waerden number will be. For example, you could say $w(k, r) \leq k^r$, but is it true? Unfortunately, it is not true because Landsman and Robertson have a theorem on pg.47 that says if $k = 3$ and $r \geq 5$, then there exists a positive constant c such that $r^{clogr} \leq w(3, r) \leq (r/4)^{3^r}$ [13] pg.47. It was an interesting conjecture, but alas there really is no pattern amongst van der Waerden numbers as the parameters get larger, which makes them so evasive.

Finding equally spaced elements in a block is not the only way arithmetic progressions can be viewed. They can also be thought of as the results of something called recurrences. [13] pg. 102; An arithmetic progression could also be viewed as the solution to the recurrence $x_k = 2x_{k-1} - x_{k-2}$. Once you get the first two values, you can get the rest of the sequence. If $x_1 = 1$ and $x_2 = 3$, then we can see $x_3 = 2x_2 - x_1 = 5$, and so on. We would use the same format if the first two values are $7, 10$, because then we can see $x_3 = 2(10) - 7 = 13$. Regardless, this is a crafty way to find the next term in a sequence that is arithmetically progressing by using equations and solutions. This becomes incredibly useful as the elements or the difference between the elements itself get very large. Instead of trying to figure out the rest of the progression in your head or using mental math to determine if the sequence is arithmetically progressing, you can simply plug the numbers into the equation above. What if your sequence was $\{108, 274, 440\}$ and you wanted to make sure this was a 3 term arithmetic sequence? Well, you could plug your numbers into the equation and make sure it works; so, we have $2(274) - 108$ which equals 440 so our sequence checks out as an arithmetic progression. Obviously, recurrences work very well with 3 term arithmetic sequence because you only need to run the recurrence equation once.

Example and Application:

Now that we know a little bit about both Ramsey's and van der Waerden's theorems within Ramsey theory, branching out further is a good way to expand this new knowledge. Ramsey's Theorem represents a way to find order of subgroups through graph theory, while Van der Waerden's Theorem represents a way to find structure through arithmetic progressions. These definitions and the example that follow reveal that with just a little variation or adjustment, sequences or sets that seem to have nothing in common may actually share commonalities or characteristics. This goes beyond Van der Waerden because this example is not an exact arithmetic progression, but it still represents how you can find structure between sets or groups that you never imagined. Remember that this paper as a whole is about how complete disorder between sets is nearly impossible and this example helps prevent that complete disorder.

Definition 5: [13] pg.183 Let $m \geq 2$ and $0 \leq a \leq m$. A k- term a (mod m)-*progression* is a sequence of positive integers $x_1 < x_2 < ...x_k$ such that $x_1 - x_{i-1} \equiv a (\text{mod } m)$ for $i = 2, 3, ...k$. This is the same concept of arithmetic progressions but you throw in the aspect of modular arithmetic. Instead of dealing with all the integers to find arithmetic progressions, you look for these progressions within Z mod n for some number n. This definition will become useful with the next example, but first here is a definition that adds to this.

Definition 6: [13] pg.183 Let $m \geq 2$. An *arithmetic progression* (mod m) is a sequence that is an a(mod m)-progression for some $a \in 1, 2, ...m - 1$. While the gaps of an arithmetic progression are all equal elements of $\mathbb{Z}$, these gaps of (mod m)-*progression* are equal when considered as elements of an additive group $\mathbb{Z}_m$.

Example: We introduced this topic of arithmetic progression in terms of modular arithmetic. Graham, Rothschild, and Spencer note that the sequence $\{1, 7, 33, 44, 70\}$ is a 5-term 1 (mod 5)-progression because when dealing with $\mathbb{Z}_5$, this sequence becomes $\{1, 2, 3, 4, 5\}$. This is interesting because at first glance of this sequence, most people would be shocked to find out there is an arithmetic progression among the elements. However, that is the beauty of working within modules because there is more than what meets the eye. Again, the sequence $\{1, 7, 33, 44, 70\}$ is only an arithmetic progression in terms of Van der Waerden's theorem if we adjust the domain of all of $\mathbb{Z}$ to $\mathbb{Z}_5$. This relates to Van der Waerden because if we adjust the theorem statement slightly, it becomes: Let $k, r \geq 2$ be integers and let us work in $\mathbb{Z}$ (mod m). There exists a least positive integer $w = w(k, r)$ such that for any $n \geq w$, every $r-$ coloring of $[1, n]$ (mod m) admits a monochromatic arithmetic progression of length k. In other words, you break up the integer n (mod m) into r classes and you should be able to find a k term arithmetic progression (mod m). Working with modular arithmetic to find arithmetic progressions represents a way to find order between sets that seem to be totally different and distinct. This is important because it is another way to verify Ramsey's main idea which is that complete order is nearly guaranteed in certain situations, and that complete disorder between sets almost can't be achieved.

Note: Working with modular arithmetic is not the only variation that can be used to help find some further commonalities between sets. [13] There are other ways you can expand or tweak Ramsey's or Van der Waerden's Theorems to find order between different sets or groups. These different ways are through such concepts as tuples, quasi-progressions, descending waves, and monotone arithmetic progression sequences or subsequences. These all represent ways to find order or commonalities between sets that appear to be totally different; after all, complete disorder is nearly impossible.

Application: There are so many applications of Ramsey's theorem but one of them is how far reaching the Pigeonhole Principle and its implications can be. [13] pg. 4 Another interesting aspect of the Pigeonhole Principle is that even though it is so simple, you can still prove something complex like guaranteeing at least two people have played the same number of opponents in Tic-Tac-Toe. Let us say T represents the set of all people who have played the game of Tic-Tac-Toe and we say there are n people in this set. Each of these n player has had at least one opponent and obviously no more than $n-1$ opponents. Place each person in T into a category based on the number of people they have played. You can see we are placing n players (pigeons) into $n-1$ categories based on their number of opponents (pigeonholes), and thus we have proved there exists at least 2 people in the same category who have played the same number of opponents in Tic-Tac-Toe. It is such a simple principle, but it clearly has far-reaching implications. This is also what I was referring to in my introduction about two of my readers having the same birthday. This is because if I had 367 readers and there are only 366 possible birthday, or subsets, in given year, at least two of my readers would share the same birthday.

Note: As I mentioned before, this is not all that Ramsey Theory has to offer. There are more than just Ramsey's and Van der Waerden Theorems within Ramsey Theory. [14] pg. 9 According Graham, there are four other important Ramsey-type theorems produced by Schur, Rado, Hales-Jewett, and Graham-Leeb-Rothschild. They all revolve around this idea of finding order or commonalities between subsets of a larger set or subgroups of a larger group. Just like Ramsey's and Van der Waerden's Theorems, these other theorems under Ramsey Theory talk about how if the initial set is big enough, then there is a better chance that the order or commonality you are looking for will be found. All of these theorems under Ramsey Theory reiterate that is more about this existence of some larger integer or group that works than it is about finding the exact lowest upper bound that allows it to occur.

Conclusion:

I wrote about the important definitions involved with Ramsey Theory, two main men behind Ramsey's Theory and their theorems, and finally I had a few useful examples and applications within this theory. Ramsey's Theory broadened my knowledge on many subjects. I used to be so naive when it came to sets that had commonalities or structure between them. After doing my research, if I see two groups or two sets with relationship or order between them, I won't just think "What a coincidence!". Instead I will try to imagine how large this superstructure or supergroup must be that they came from; from there, my mind will go to a place of establishing an upper or lower bound on that larger set to determine if we could make it smaller yet still contain this relationship when broken up. Basically, my mind will never be the same when it comes to finding structure or order between two sets or groups, and I can thank Ramsey Theory for that. Ramsey Theory revolves around how if a set of group has a particular property and gets broken down into subgroups of subsets, then one of those subgroups or subsets is guaranteed to have that same property. Working back the other way though is if you see certain subgroups have this property, the question becomes how small can you make the larger group where the subgroups will still have this property. The most crucial detail about Ramsey Theory is not that you can always find the exact Ramsey or Van der Waerden number given certain parameters, but that you know there is some larger graph or integer that definitely allows it to occur. It relates back to this main idea that complete order between sets can't be promised, and that complete disorder between them is close to unattainable. In the end, the most important thing about learning something new though is determining the applications of this new found information. For starters, if you were throwing a party and wanted to make sure there would be a certain numbered group of mutual friends or mutual strangers, then Ramsey would tell you that there definitely is a certain number of guests that allows these subgroups to occur. Also, it is quite impressive that you are able to tell someone that there definitely exists an integer that you could write out such that specific colorings through Van der Waerden will always produce arithmetic progressions of a particular length. Overall, Ramsey Theory has altered the way I look at sets or groups which have commonalities between them.

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