Proving Napoleon's Theorem

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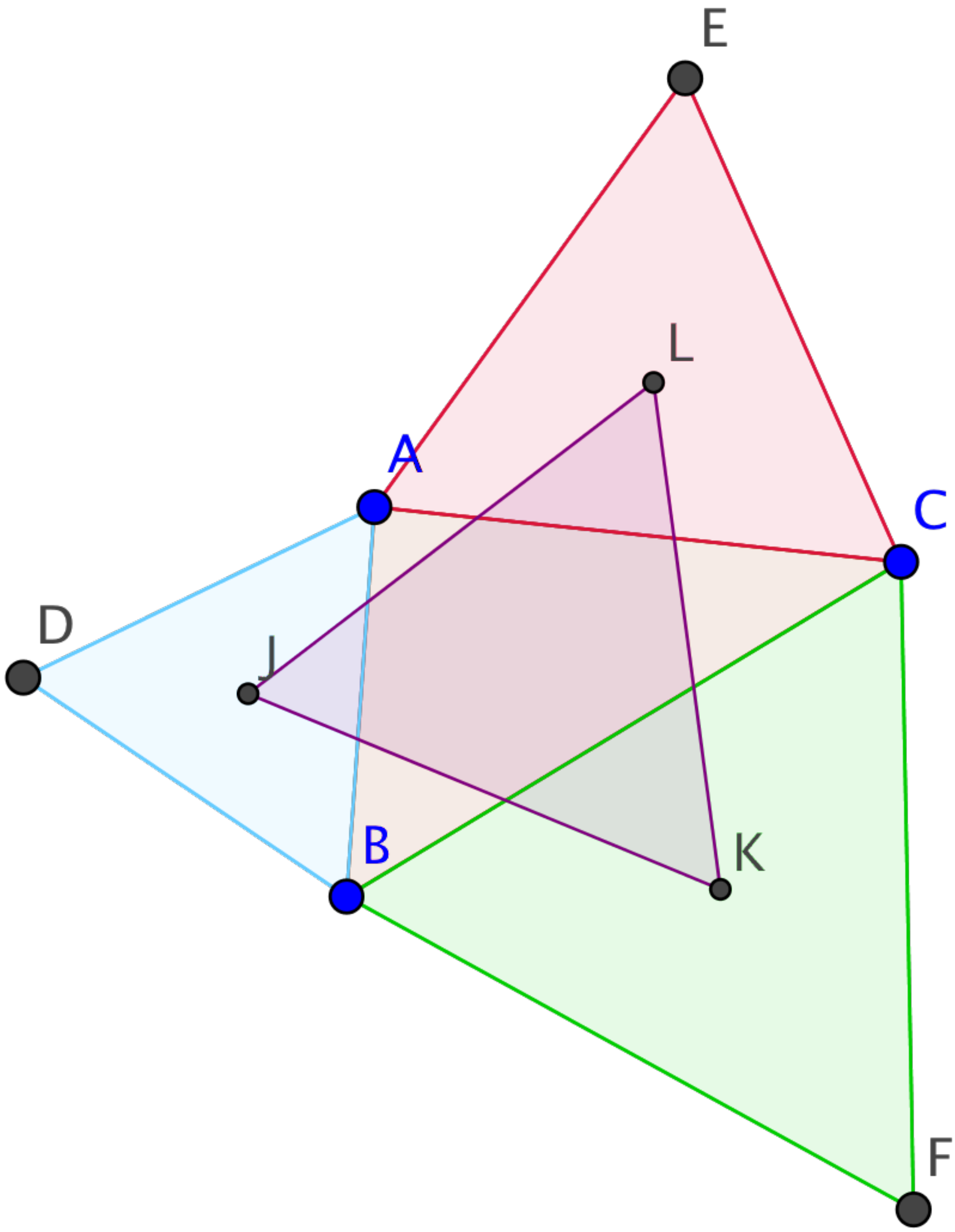

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1 Introduction

Napoleon Bonaparte (1769-1821) was known for his many achievements within his lifetime. Throughout his career as a military leader, Bonaparte managed to conquered most of Europe and crowned himself the French Emperor. However, it is a little known fact that Napoleon had a keen interest in mathematics as well. He was known to be acquainted with many well-known mathematicians at the time, constantly engaging in conversations about geometric structures. Thus, it was because of this interest in geometry that rumors circulated that Napoleon was the founder of a very important geometric theorem which states [1]

Theorem 1 (Napoleon's)**. *Describe equilateral triangles (the vertices being either all inward,) upon the three sides of any given triangle ABC: then the lines which join the centres of gravity of those three equilateral triangles will constitute an equilateral triangle.***

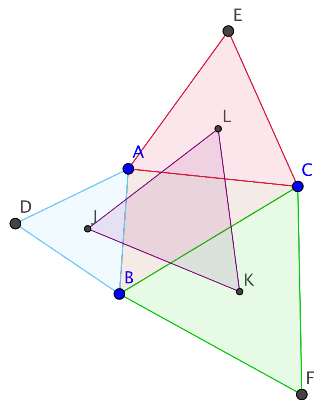

Figure 1: **Napoleon's Theorem**
In the graphic above, let $\triangle ABC$ be our arbitrary triangle. Thus when we construct the equilateral triangles $\triangle ACE, \triangle CBF$, and $\triangle DAB$ on the sides of $\triangle ABC$ and connect their centers L, K, and J respectively, we form the equilateral triangle $\triangle LKJ$.

While Napoleon was initially credited for this theorem, there was a bit of controversy surrounding this claim. Napoleon's Theorem was initially published in *The Ladies' Diary* in 1825, four years after Napoleon's death, by William Rutherford. Rutherford, a famous English mathematician, never mentioned Napoleon in his initial printing and, in fact, Napoleon's name didn't even appear in a publication relating to the theorem until 1911. However, with the attachment of the French leader's name, the theorem sky-rocketed in popularity and since then has just been widely accepted as the theorem belonging to Napoleon [1]. All controversy aside, since its inception, Napoleon's theorem has become the foundation of many elementary geometric theorems and generalizations. Multiple mathematicians have individually re-discovered Napoleon's theorem and have observed interesting properties that these Napoleonic formations possessed [1]. In this paper, we will go over four different proofs of Napoleon's Theorem as well as the properties that arise from each of them. We will then discuss two generalizations of Napoleon's theorem that have a lasting impression on the formations of different n-gons.

2 Preliminary Definitions

This section of the paper contains basic definitions to further understanding of Napoleon's Theorem.

Definition 1. *A **median** of a polygon is a line segment that joins a vertex to the midpoint of the opposing side.*

Definition 2. *The **centroid** of a polygon is the point where all medians intersect. It is also referred to as the **centre**.*

Definition 3. *An **affine** transformation is any transformation that preserves collinearity and ratios of distances.*

Definition 4. *The **circumscribed circle** or **circumcircle** of a polygon is a circle which passes through all the vertices of the polygon. The center of a circumcircle is called the **circumcenter**.*

Definition 5. *The vertex of an isosceles triangle that has an angle different form the two equal angles is called the **apex** of the isosceles triangle. The angle that defines the apex of the isosceles triangle is called the **apex angle**.*

3 Proofs

3.1 A Trigonometric Proof of Napoleon's Theorem

Proof. [2] Suppose we have an arbitrary triangle ABC. Let us denote the respective centroids of the outward-facing equilateral triangles on sides BC, CA, and AB by D, E, and F. Recall that for a triangle with sides of length a, b, and c, we can find the length of the sides using the cosine rule which states

$$c^2 = a^2 + b^2 - 2ab\cos C \tag{1}$$

where C is the angle between sides a and b.

Notice how $\angle EAC$ is equivalent to the $\angle FAB$ which has measure $\frac{\pi}{3}$. Thus by using the law of cosines on the triangle formed by the centroids DEF, we get the following:

$$EF^2 = AE^2 + AF^2 - 2(AE)(AF)\cos(A + \frac{\pi}{3}) \tag{2}$$

Since the centroids are along the medians, they are two-thirds the distance from the vertex to the midpoint of the opposite side. Hence,

$$AF = c\left(\frac{2}{3}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{c\sqrt{3}}{3}\left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \frac{c}{\sqrt{3}} \tag{3}$$

$$AE = b\left(\frac{2}{3}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{b\sqrt{3}}{3}\left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \frac{b}{\sqrt{3}}. \tag{4}$$

Therefore, by applying these equations to Equation 2, we result with the following:

$$
\begin{aligned}
EF^2 &= \left(\frac{b}{\sqrt{3}}\right)^2 + \frac{c}{\sqrt{3}}^2 - 2\left(\frac{b}{\sqrt{3}}\right)\left(\frac{c}{\sqrt{3}}\right)\cos(A+\frac{\pi}{3}) \\
&= \frac{b^2}{3} + \frac{c^2}{3} - 2\left(\frac{bc}{3}\right)\cos(A+\frac{\pi}{3}) \\
3EF^2 &= b^2 + c^2 - 2bc\cos(A+\frac{\pi}{3}).
\end{aligned} \tag{5}
$$

Recall that the cosine of the sum of two angles α and β is

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta. \tag{6}$$

Recall that $\cos(\frac{\pi}{3}) = \frac{1}{2}$ and $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$. Hence,

$$
\begin{aligned}
\cos(A+\frac{\pi}{3}) &= \cos A\cos(\frac{\pi}{3}) - \sin A\sin(\frac{\pi}{3}) \\
\cos(A+\frac{\pi}{3}) &= \frac{1}{2}\cos A - \frac{\sqrt{3}}{2}\sin A.
\end{aligned} \tag{7}
$$

Substituting Equation 7 into Equation 5, we get the following:

$$
\begin{aligned}
3EF^2 &= b^2 + c^2 - 2bc(\frac{1}{2}\cos A - \frac{\sqrt{3}}{2}\sin A) \\
&= b^2 + c^2 - bc(\cos A - \sqrt{3}\sin A) \\
&= b^2 + c^2 - bc\cos A + \sqrt{3}bc\sin A.
\end{aligned} \tag{8}
$$

We can further expand this equation by applying the Law of Cosines to our arbitrary triangle ABC,

$$
\begin{aligned}
a^2 &= b^2 + c^2 - 2bc\cos A \\
a^2 - b^2 - c^2 &= -2bc\cos A \\
\frac{1}{2}(a^2 - b^2 - c^2) &= -bc\cos A.
\end{aligned} \tag{9}
$$

Now applying the Law of Sines, we can expand the equation even further;

$$
\begin{aligned}
\frac{a}{sinA} &= \frac{b}{\sin B} = \frac{c}{\sin C} \\
a &= \frac{b\sin A}{\sin B} = \frac{c\sin A}{\sin C} \\
a &= \\
2\Delta &= bc\sin A.
\end{aligned} \tag{10}
$$

where Δ is the area of ABC.

We can now apply Equations 9 and 10 to Equation 8 to get the following result,

$$
\begin{aligned}
3EF^2 &= b^2 + c^2 + \frac{1}{2}(a^2 - b^2 - c^2) + 2\sqrt{3}\Delta \\
3EF^2 &= b^2 + c^2 + \frac{a^2}{2} - \frac{b^2}{2} - \frac{c^2}{2} + 2\sqrt{3}\Delta \\
3EF^2 &= \frac{a^2}{2} + \frac{b^2}{2} + \frac{c^2}{2} + 2\sqrt{3}\Delta \\
EF^2 &= \frac{a^2}{6} + \frac{b^2}{6} + \frac{c^2}{6} + \frac{2\sqrt{3}}{3}\Delta.
\end{aligned} \tag{11}
$$

Thus, by symmetry, we have $EF = FD = DE$ as required.

□

3.2 An Analytical Proof of Napoleon's Theorem

Proof. [3] Suppose we have an arbitrary triangle. We shall give the vertices of the triangle the cartesian coordinates (x_i, y_i) where the index $i = 0, 1, 2$ and assign them consecutively from any one vertex as a starting point in a counterclockwise progression.

To find the coordinates (a, b) of the centroid, c, of the exterior (or interior) triangle attached to the segment connecting (x_i, y_i) to (x_{i+1}, y_{i+1}), consider the right triangle formed on the interior (or exterior if your equilateral triangles are formed on the interior of the arbitrary triangle) of the the segment between (x_i, y_i) and (x_{i+1}, y_{i+1}). The coordinates of the apex of this interior right triangle will be $(\Delta x, \Delta y)$. The angle formed by $c, (x_i, y_i)$ and (x_{i+1}, y_{i+1}) will be $\frac{\pi}{6}$ and let the angle formed by $(x_{i+1}, y_{i+1}), (x_i, y_i)$, and $(\Delta x, \Delta y)$ be called θ.

Now consider another interior right triangle formed with c and (x_i, y_i). The angle of (x_i, y_i) will be $\tan(\frac{\pi}{6} + \theta)$. Thus, using the sum property of tan, we get the following;

$$
\begin{aligned}
\tan(\frac{\pi}{6} + \theta) &= \frac{\tan(\frac{\pi}{6}) + \tan(\theta)}{1 - \tan(\frac{\pi}{6})\tan(\theta)} \\
&= \frac{\frac{1}{\sqrt{3}} + \frac{\Delta x}{\Delta y}}{1 - \frac{1}{\sqrt{3}}(\frac{\Delta x}{\Delta y})} \\
&= \frac{\sqrt{3}\Delta y - \Delta x}{\sqrt{3}\Delta y}
\end{aligned}
$$

$$
\tan(\frac{\pi}{6} + \theta) = \frac{\tan(\frac{\pi}{6}) + \tan(\theta)}{1 - \tan(\frac{\pi}{6})\tan(\theta)}
$$

Notice that with the addition of our interior right triangle, the coordinates of c are

$$
(a - x_i, b - y_i)
$$

The centroid of the exterior (or interior) triangles attached to the side connecting (x_i, y_i) to (x_{i+1}, y_{i+1}) lies at the following coordinate:

$$
\begin{pmatrix} xc_i \\ yc_i \end{pmatrix} = \begin{pmatrix} \frac{x_i + x_{i+1}}{2} \pm \frac{y_i + y_{i+1}}{2\sqrt{3}} \\ \frac{y_i + y_{i+1}}{2} \pm \frac{x_i + x_{i+1}}{2\sqrt{3}} \end{pmatrix}
$$

Thus the square of the distance between (xc_i, yc_i) to (xc_{i+1}, yc_{i+1}) can be described as

$$\frac{1}{3}[x_i^2 + x_{i+1}^2 + x_{i+2}^2 - x_i x_{i+1} - x_{i+1} x_{i+2} - x_{i+2} x_i]$$

for the x coordinates,

$$\frac{1}{3}[x_i^2 + x_{i+1}^2 + x_{i+2}^2 - x_i x_{i+1} - x_{i+1} x_{i+2} - x_{i+2} x_i]$$

for the y coordinates, and a third quadratic

$$\pm\frac{1}{\sqrt{3}}[x_i(y_{i+1} - y_{i+2}) + x_{i+1}(y_{i+2} - y_i) + x_{i+2}(y_i - y_{i+1})]$$

which consists of x and y coordinates. Under inspection, none of these quadratics changes as the index i increases, which shows that the centroid locations occupy the vertices of an equilateral triangle. □

3.3 A Proof of Napoleon Theorem Involving Complex Numbers

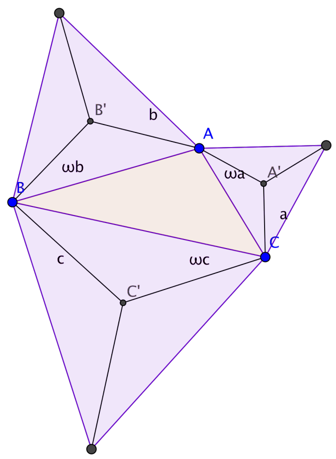

Figure 2: We will be using this figure to prove Napoleon's Theorem with complex numbers.

Before starting this proof, we will briefly go over Euler's Formula. Euler's Formula states that for any angle θ,

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

The value $e^{i\theta}$ is a unit complex number which traces the unit circle in the complex plane. Therefore, multiplication by $e^{i\theta}$ results in a rotation through a positive angle of θ. With this stated, we can begin our proof.

Proof. [4] In the complex plane, let the triangles ABC and $A'B'C'$ be formed with vectors a, b, c and $\omega = e^{\frac{i\pi}{3}}$, so that $\omega^3 = -1$ and $\omega^2 + 1 = \omega$. Therefore,

$$\overrightarrow{AB} = (1+\omega)c,$$
$$\overrightarrow{BC} = (1+\omega)a,$$
$$\overrightarrow{CA} = (1+\omega)b.$$

Similarly,

$$\overrightarrow{A'B'} = \omega a + b,$$
$$\overrightarrow{B'C'} = \omega b + c,$$
$$\overrightarrow{C'A'} = \omega c + a.$$

Notice how

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = (1+\omega)(a+b+c) = 0.$$

This implies that $a + b + c = 0$. It is important to note that $A'B'C'$ is an equilateral triangle if and only if

$$\omega(\overrightarrow{A'B'}) = \overrightarrow{A'C'}$$
and
$$\omega(\overrightarrow{C'A'}) = \overrightarrow{B'C'}$$

Therefore,

$$\begin{aligned}\omega(\overrightarrow{A'B'}) - \overrightarrow{A'C'} &= \omega(\omega a + b) + (\omega c + a) \\ &= (\omega^2 + 1)a + \omega b + \omega c \\ &= \omega(a + b + c) = 0\end{aligned}$$

Similarly, $\omega(\overrightarrow{C'A'}) - \overrightarrow{B'C'} = 0$ as well. Thus, $A'B'C'$ is indeed an equilateral triangle. □

4 Properties of Napoleonic Triangles

4.1 Fermat Points

In a letter to fellow mathematician Evangelista Torricelli (1608-1647), French lawyer and mathematician Pierre de Fermat challenged Torricelli to solve the following problem: "Find the point such that the sum of its distances from the vertices of a triangle is a minimum." Since then, many mathematicians have come up with their own solution to the problem. For example, a simple solution was founded by Joseph Ehrenfried Hofmann(1900-1973) in 1929. It states the following:

Solution 1 (Hofmann)**.** *[6] In $\triangle ABC$, select point P and connect it with vertices A, B, and C. Rotate $\triangle ABP 60°$ around B into position $C'BP'$. By construction, $\triangle BPP'$ is equilateral. Also notice that,*

$$PB = P'B, and PA = C'P'.$$

Therefore,

$$PA + PB + PC = C'P + P'P + PC.$$

Notice that the broken line $CPP'C'$ is no shorter than the straight line CC'. Hence,

$$PA + PB + PC \geq CC'$$

Therefore, $PA + PB + PC$ reaches its minimum if and only if P lies on CC'. Similarly, had we rotated $\triangle ABP$ around A, we would have found that

$$PA + PB + PC \geq BB',$$

and rotating $\triangle ABP$ around C would lead us to conclude that

$$PA + PB + PC \geq AA'.$$

Since $PA + PB + PC$ is minimized when P lies on AA', BB', and CC' we can conclude that P is minimized at the intersection of those three segments.

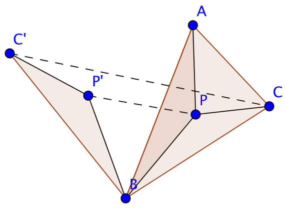

Figure 3: Hofmann's solution.

This point within a triangle became known as a Fermat point. It is also known as a Fermat-Torricelli point to some.

Definition 6. *A **Fermat point**, or **Fermat-Torricelli point**, of a triangle is the point such that the sum of its distances from all three vertices is a minimum.*

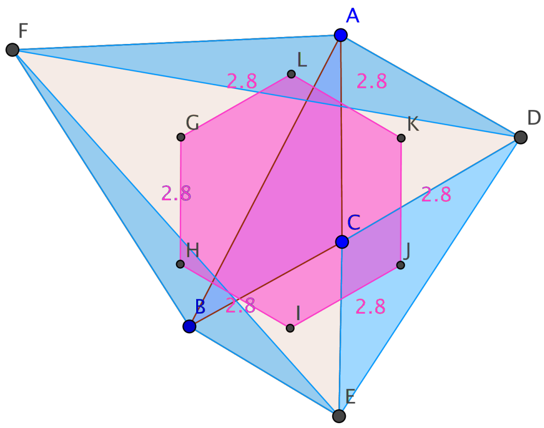

Figure 4: A Napoleonic hexagon can be formed by connecting the Fermat points of triangles AFD, CDE, FBE with the centroids of the equilateral triangles formed on the side of ABC.

[5] By finding the Fermat point of Napoleonic triangles, we notice a property that all Napoleonic triangles share.

Definition 7. *Two lines passing through a vertex of a triangle are called **isogonal** with respect to that vertex if they form congruent angles with its (internal) angle bisector.*

Theorem 2. *[5] Construct an arbitrary triangle ABC and create points A', B', and C' such that lines AB' and AC' are isogonal as are pairs CB', CA' and BA', BC'. Then the lines AA', BB', and CC' meet at a common point.*

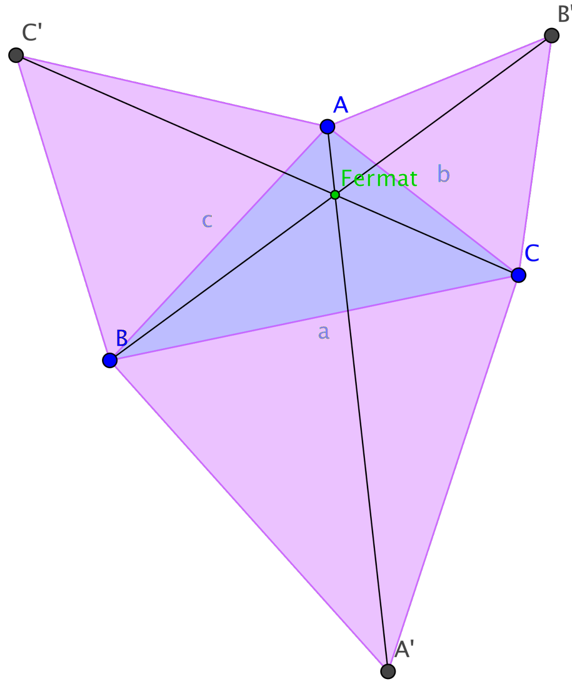

Figure 5: The Fermat point in a Napoleonic formation is the intersection of line AA', BB', and CC'.

Notice that Napoleon's theorem is true when the measurements of all three angles involved are equal to 30°. Thus, we get the following property:

Property 1. *[5] Let $\triangle ABC_o, \triangle BCA_o$, and $\triangle CAB_o$ be the outer equilateral triangles and $\triangle ABC_i, \triangle BCA_i$, and $\triangle CAB_i$ be the inner equilateral triangles. Then the midpoints of AA_o and B_iC_i coincide as does the midpoints of BB_o, C_iA_i, and CC_o, A_iB_i. A similar assertion holds with the exchange of the indices o and i throughout.*

4.2 Napoleonic Triangles and Circumcircles

Another property of Napoleonic triangles can be observed by looking at Fermat points in relation to circumcircles. Another version of Napoleon's theorem can be stated as the following:

Theorem 3 (Napoleon's (Version 2))**.** *[5] On each side of a triangle, erect an equilateral triangle, lying exterior to the original triangle. Then the segment connecting the circumcenters of the three equilateral triangles themselves form an equilateral triangle.*

It is also worth noting the following corollary:

Corollary 1. *[5] If similar triangles $\triangle PCB, \triangle CQA$ and $\triangle BAR$ are erected externally on the sides of a triangle $\triangle ABC$, their circumcenters form a triangle similar to the given three triangles.*

With both the rewritten Napoleon's theorem and the previous corollary in mind, we can observe the following property of Napoleonic triangles:

Property 2. *[5] The sides of a Napoleonic triangle serve as perpendicular bisectors of the lines joining the Fermat point to the vertices of* $\triangle ABC$*.*

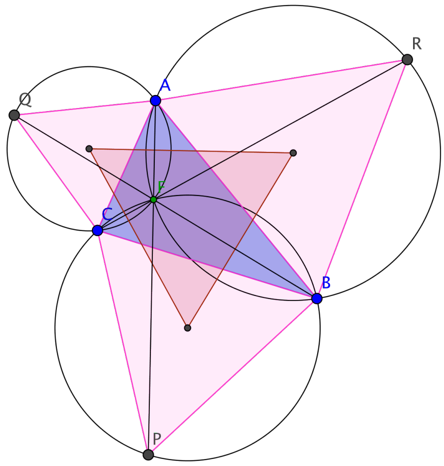

Figure 6: The sides of the Napoleonic triangle are perpendicular bisectors for the lines joining the Fermat point, F, to the vertices of $\triangle ABC$.

This property comes from the fact that by the construction of the Napoleonic triangle, the vertices are the circumcenters of the circumcircles of $\triangle ABC''$, $\triangle AB''C$, and $\triangle A''BC$ while the segments AF, BF, CF, where F is the Fermat point, are the common chords of the circles taken two at a time.

4.3 Midpoint Reciprocity in Napoleonic Triangles

Another property that Napoleonic triangles have has to do with the formation of exterior and interior equilateral triangles on our original triangle and their midpoints. This property states the following:

Property 3. *[3] Let* $\triangle ABC_o, \triangle BCA_o, \triangle CAB_o$ *be the outer Napoleon's triangles and* $\triangle ABC_i, \triangle BCA_i, \triangle CAB_i$ *be the inner Napoleon's triangles. Then the midpoints of* AA_o *and* B_iC_i *coincide as are the midpoints of* BB_o, C_iA_i*, and* CC_o, A_iB_i*. A similar assertion holds with the exchange of the indices o and i throughout.*

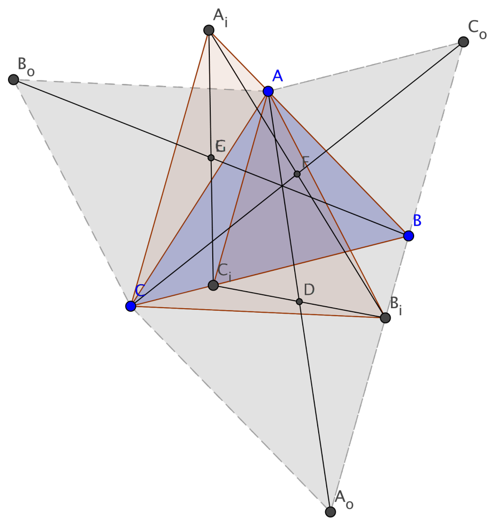

Figure 7: The Midpoint Reciprocity in Napoleonic Triangles.

This property is justified using the following proof:

Proof. Let $\omega = \cos 60° + i \sin 60°$ be a counterclockwise rotation through 60°. Treating points as complex numbers,

$$A_o = C + (B - C)\omega$$
$$B_i = C + (A - C)\omega$$
$$C_i = A + (B - A)\omega,$$

such that

$$\frac{A + A_o}{2} = \frac{(A + C) + (B - C)\omega}{2}.$$

Also,

$$\frac{B_i + C_i}{2} = \frac{(A + C) + (B - C)\omega}{2}.$$

Thus, the midpoints are equivalent. The results are similar for the midpoints of the other segments. □

5 Generalizations and Extensions

5.1 Napoleon-Barlotti Theorem

This generalization of Napoleon's theorem was developed by Italian mathematician Adriano Barlotti in 1955. Barlotti discovered this generalization by observing both Napoleon's theorem and another theorem named Thébault's Theorem which states the following:

Theorem 4 (Thébault)**.** *[8] On the sides of a parallelogram ABCD erect squares - all either on the exterior or interior of the parallelogram. The centeroids of the squares form another square.*

The similarities between Thébault's and Napoleon's theorem are obvious, yet Barlotti decided to generalize these geometric theorems even further by extending them to apply to arbitrary n-gons.

Theorem 5 (Napoleon-Barlotti). *On the sides of an affine regular n-gon, construct regular n-gons (all on the exterior or all on the interior). Then the centroids of these regular n-gons form a regular n-gon.*

By "regular n-gon" we understand any polygon of n-sides such that symmetries act transitively on its vertices and on its edges; thus this expands the theorem to apply not only to convex polygons.

5.2 Petr-Douglas-Neumann Theorem

The Petr-Douglas-Neumann theorem [9] has a complex history. The theorem, often referred to as the PDN theorem, was first published in 1905 by Czech mathematician Karel Petr. However, for a long time, it was believed that the theorem was founded by American mathematician Jesse Douglas and British-Australian mathematician Bernhard Neumann since they were the first to publish it in common diction. (Petr published the theorem in Czech.) The theorem itself generalizes Napoleon's theorem to apply to different polygons and shows that regular polygons can be formed by using the sides of an initial polygon.

Theorem 6 (Petr-Douglas-Neumann). *[9] If isosceles triangles with apex angles $\frac{2k\pi}{n}$ are erected on the sides of an arbitrary n-gon A_0, and if this process is repeated with the n-gon formed by the apexes of the newly formed triangles but with a different value of k and so on until all values $1 \leq k \leq n-2$ have been used (in arbitrary order), then a regular n-gon A_{n-2} is formed whose centroid coincides with the centroid of A_0.*

Definition 8. *An operator L is said to be* ***linear operator*** *if, for every pair of functions f and g and a scalar t,*

$$L(f+g) = Lf + Lg$$

and

$$L(tf) = t(Lf).$$

Definition 9. *A* ***linear functional*** *on a real vector space V is a function $T : V \to R$, which satisfies the following properties:*

1. For all v, w in V

$$T(v+w) = T(v) + T(w).$$

2. For a scalar α,

$$T(\alpha v) = \alpha T(v).$$

Proof. Represent an n-gon by the list of its vertices thought of as complex numbers, that is, by a vector in C^n. Suppose there is an n-gon A and an n-gon B formed by the free vertices of similar triangles built on the sides of A. For a fixed complex number α determining the shape of the similar triangles,

$$\alpha(A_i - B_i) = A_{i+1} - B_i, \tag{12}$$

that is,

$$B_i = (1-\alpha)^{-1}(A_{i+1} - \alpha A_i).$$

Now consider the linear operator $S : C^n \to C^n$ that cyclically permutes the coordinates one place. Equation 13 then transforms into

$$B_i = (1 - \alpha)^{-1}(S - \alpha I)A \tag{13}$$

where I is the identity. Taking centroids is a linear functional on this space of n -gons and is invariant under S, so we see that B has the same centroid as A. This implies that the polygon A_{n-2} is obtained from A_0 by applying the composition of the operators

$$(1 - \omega^k)^{-1}(S - \omega^k I) \text{ for } k = 1, 2, ..., n - 2$$

where $\omega = e^(2\pi i/n)$.

To proceed we need a criterion for when a polygon is regular. P is a regular n-gon if each side is obtained from the next by rotating an angle of $2\pi i/n$, that is, if

$$P_{i+1} - P_i = \omega(P_{i+2} - P_{i+1}).$$

In terms of the S, P is regular if it is in the kernel of

$$(S - I)(I - \omega S) \text{ or } (S - I)(S - \omega^{n-1} I)$$

Since, $S^n - I = 0$, the composition of all the $S - \omega^k I$ for $k = 0, 1, ..., n - 1$ is zero, thus proving the theorem. □

6 Conclusion

Napoleon's theorem is one of the most rediscovered results in mathematics. [10] There are many variations of Napoleon's theorem which stem from iterating its basic construction (constructing regular polygons on the sides of an arbitrary polygon) with different n-gons and observing different properties that occur within the formations. Other uses of Napoleonic formations includes forming tessellations and architecture.

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