Saint Mary's College of California

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Senior Essay

Banach Spaces

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May 21, 2017

1 History

Functional Analysis is the field of mathematics specializing in the study of certain types of functions between vector spaces and their scalar fields. The beginnings of Functional Analysis can be traced back to the 20th century Polish mathematician Stefan Banach [11](March 30, 1892-August 31, 1945). Born in Krakow in modern day Poland, Banach studied engineering at Lvov Technical University before going on to earn his doctorate in Mathematics from Lvov in 1920. His thesis *On Operations on Abstract Sets and their Applications to Integral Equations* "is sometimes said to mark the birth of functional analysis." In his thesis he described what would later be referred to as "Banach spaces." These spaces have become indispensable in the study of modern analysis and physics.

In this essay we will work our way through some of the most necessary concepts that Banach spaces are built on, examine some of the differences between finite and infinite dimensional spaces, and then look at the Hahn-Banach Theorems which answers several of the most important questions of Functional Analysis of the early twentieth century. At the end we will look at the consequences and applications of the Hahn-Banach Theorems. Hans Hahn [11](1879-1943), an Austrian mathematician, and Stefan Banach both deduced the statement of the Hahn-Banach Theorem within two years of each other, Hahn in 1927 and Banach in 1929, but neither was aware of the other's work and they used different approaches to prove this fundamental theorem in Functional Analysis.

Figure 1: Hans Hahn

Figure 2: Stefan Banach

2 Normed Linear Spaces

Linear spaces, also known as vectors spaces, can be thought of as a collection of objects called "vectors" that can be added together and multiplied by numbers called "scalars." They are a widely used mathematical concept and are fundamental to the study of Physics. We begin this section with some definitions about linear spaces, their properties, and some examples.

Definition[6]: A *linear space* or *vector space*, $V = V(\mathbb{K})$, over a field, $\mathbb{K}$, is an algebraic structure equipped with two operations called vector addition and scalar multiplication such that for any *vectors*, $u, v \in V$, and any *scalars*, $\alpha, \beta \in \mathbb{K}$ the following properties hold:

1. $u + v \in V$ (Closure of Addition)
2. $(u + v) + w = u + (v + w)$ (Associativity of Addition)
3. $u + v = v + u$ (Commutativity of Addition)
4. $\alpha\beta u = \beta\alpha u$ (Commutativity of Multiplication)
5. There exists an element, $\mathbf{0} \in V$, such that for all $u \in V$, $\mathbf{0} + u = u$. (Additive Identity)
6. For all $u \in V$, there exists an element $-u \in V$ such that $u + (-u) = \mathbf{0}$. (Closure under Additive Inverses)
7. For every $\alpha \in \mathbb{K}$ and $u \in V$, $\alpha u \in V$. (Closure under Multiplication)
8. There exists an element $1 \in \mathbb{K}$ such that for any $u \in V$, $1u = u$. (Multiplicative Identity)
9. $(\alpha + \beta)u = \alpha u + \beta u$ (Distributivity)
10. $\alpha(u + v) = \alpha u + \alpha v$ (Distributivity)
11. $(\alpha\beta)(u) = \alpha(\beta u) = \beta(\alpha u) = \alpha\beta u$ (Associativity of Multiplication)
12. There exists an element $0 \in \mathbb{K}$ such that $0u = \mathbf{0}$, for all $u \in V$. (Zero Multiplicative Element/Absorptivity)

In this essay all linear spaces are assumed to be over the real numbers, $\mathbb{R}$.

Some Examples of Normed Linear Spaces are:

1. One example well known to all Linear Algebra students is the set $\mathbb{R}^2$ with the usual operations of addition " $+$ " and scalar multiplication: For all $a = (a_1, a_2), b = (b_1, b_2) \in \mathbb{R}^2$ and $c \in \mathbb{R}$.

(i) $a + b = (a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$, and

(ii) $ca = c(a_1, a_2) = (ca_1, ca_2)$.

We can visualize these operations by looking at each point on the Euclidean plane $\mathbb{R}^2$ as a vector. In fact in figure 3 on the next page we can see that the addition is commutative. From figure 4 on the next page, scalar multiplication can be understood as the lengthening or shortening of a vector. In the case that the scalar is negative, the direction of the vector is reversed.

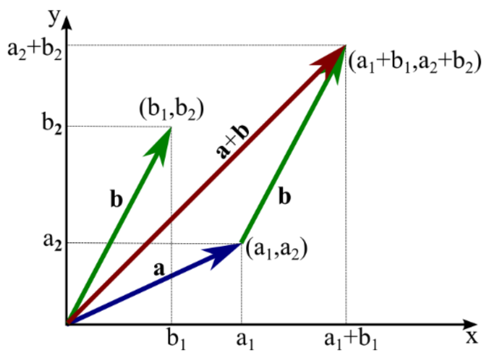

Figure 3: vector addition in $\mathbb{R}^2$

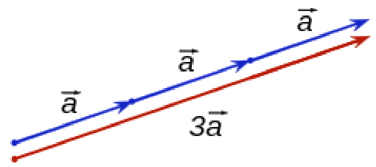

Figure 4: scalar multiplication in $\mathbb{R}^2$

2. We can extend the above example to $\mathbb{R}^n$ for any natural number n where the elements of $\mathbb{R}^n$ are n-tuples; i.e., $a = (a_1, a_2, \dots, a_n)$.

3. The set $\mathbb{R}[X]$ of all polynomials with real coefficients with polynomial addition as the vector addition and scalar multiplication as multiplying each term by a real number.

4. The set $C([a, b])$ of all continuous real-valued functions defined on the interval $[a, b]$ with addition being the sum of functions and scalar multiplication being the multiplication of a function by a real number.

We now turn to define a new space from a linear space V by giving V a real-valued operation called a norm.

Definition[6]: A *norm* on a linear space V is a function $\|\cdot\| : V \to \mathbb{R}$ such that for all $x, y \in V$ and $\alpha \in \mathbb{R}$,

1. $\|x\| \geq 0$. (Non-negativity)
2. $\|x\| = 0$ if and only if $x = 0$.
3. $\|\alpha x\| = |\alpha| \|x\|$. (homogeneity)
4. $\|x + y\| \leq \|x\| + \|y\|$. (triangle inequality)

Definition[10]: A *normed linear space* is a linear space V along with a norm $\|\cdot\| : V \to \mathbb{R}$. We denote the normed linear space as a pair $(V, \|\cdot\|)$.

The norm formalizes the notion of finding the "length" of a vector. Some concepts that are related to the norm are convergence, continuity, special neighborhoods of points called open balls, and distance. We will formalize and discuss these topics below.

Definition[10]: Let $(V, \|\cdot\|)$ be a normed linear space. We say that a sequence $\{x_n\}_{n=1}^{\infty}$ in V *converges* to $x \in V$ provided that

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

Definition[6]: A function $f : X \to \mathbb{R}$ is *continuous* at $x \in X$ provided that if for each sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ such that x_n converges to x, then $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x) \in \mathbb{R}$.

Theorem 1 (6)**.** *The norm is a continuous function.*

Proof. Let $\{x_n\}$ be a sequence in V such that x_n converges to $x \in V$. Then

$\|x_n\| = \|x_n - x + x\| \leq \|x_n - x\| + \|x\|$. Thus, $\|x_n\| - \|x\| \leq \|x_n - x\|$

and, $\|x\| = \|x - x_n + x_n\| \leq \|x - x_n\| + \|x_n\| = \|x_n - x\| + \|x_n\|$.

This gives us, $\|x\| - \|x_n\| \leq \|x_n - x\|$ or $\|x_n\| - \|x\| \geq -\|x_n - x\|$.

Therefore, $0 \leq \big|\|x_n\| - \|x\|\big| \leq \|x_n - x\|$.

Since $\lim_{n\to\infty} \|x_n - x\| = 0$ by assumption, we have $\lim_{n\to\infty} \big|\|x_n\| - \|x\|\big| = 0$, so the norm is a continuous function. □

Some examples of norms are:

1. the absolute value, $||x|| = |x|$ for all $x \in \mathbb{R}$
2. the Euclidean norm, $||x|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ for all $x \in \mathbb{R}^n$
3. the magnitude of a complex number, $||z|| = \sqrt{Re(z)^2 + Im(z)^2}$

A linear space can be given many different norms. Sometimes, the normed linear spaces with different norms are equivalent. We will now look at how we can formalize this notion and show how they are equivalent.

Definition[6]: In a normed linear space, V, with norm, $||\cdot||_n$, the *open unit ball with center* 0 *and radius* r is given by

$$B_n(0,r) = \{x \in V : ||x||_n < 1\}$$

Definition[10]: Let V be a normed linear space with two norms $||\cdot||_1$ and $||\cdot||_2$. We say that these two norms are *equivalent* if and only if there exists some constants $c > d > 0$ such that

$$B_1(0,d) \subseteq B_2(0,1) \subseteq B_1(0,c)$$

Consider $V = \mathbb{R}^2$ with the following norms:

- $||x||_1 = |x_1| + |x_2|$
- $||x||_2 = \sqrt{x_1^2 + x_2^2}$
- $||x||_3 = max\{|x_1|, |x_2|\}$

Here is the open unit ball for each of these norms on $\mathbb{R}^2$:

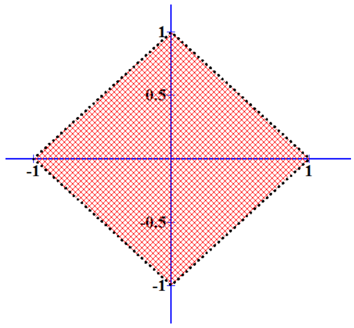

Figure 5: $||x||_1 = |x_1| + |x_2|$

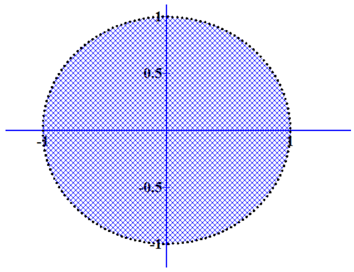

Figure 6: $||x||_2 = \sqrt{x_1^2 + x_2^2}$

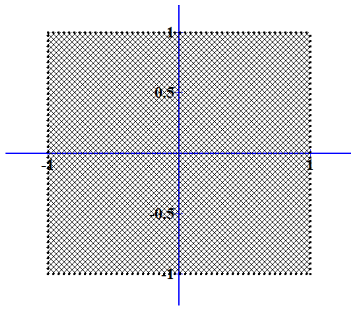

Figure 7: $||x||_3 = max\{|x_1|, |x_2|\}$

These three norms can be seen to be equivalent in $\mathbb{R}^2$ because in the first illustration below we can shrink the open unit ball for norm 1 inside the open unit ball for norm 2, and we can also enlarge the open unit ball from norm 1 so that the open unit ball in norm 2 lies inside it. This is also true for norm 2 and 3.

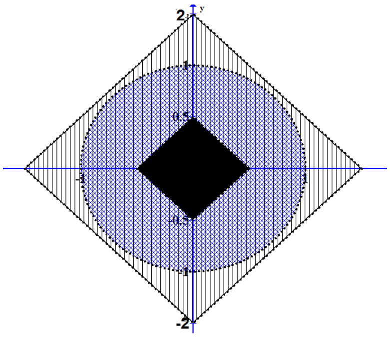

Figure 8: $\|x\|_1 \cong \|x\|_2$

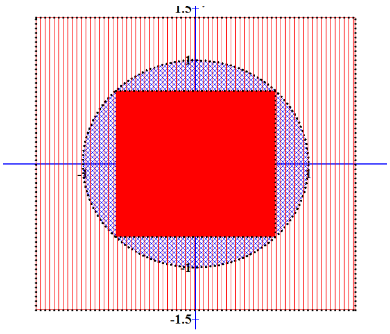

Figure 9: $\|x\|_2 \cong \|x\|_3$

So by transitivity $\|x\|_1 \cong \|x\|_2 \cong \|x\|_3$.

On a normed linear space, it is often useful to formalize the concept of "distance" in the space. This formalization is given by a function called a metric.

Definition[6]: A *metric* on a non-empty set X is a function $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$:

1. $d(x, y) \geq 0$ (non-negativity)
2. $d(x, y) = 0$ iff $x = y$
3. $d(x, y) = d(y, x)$ (symmetry)
4. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

A non-empty set with a metric is called a *metric space*.

Proposition 1. *Let $(V, \|\cdot\|)$ be a normed linear space. Then the function $d : V \times V \to \mathbb{R}$ defined by $d(x, y) = \|x - y\|$ is a metric on V.*

Proof. Let $(V, \|\cdot\|)$ be a normed linear space. Let $x, y, z \in V$. Define $d : V \times V \to \mathbb{R}$ by

$$d(x, y) = \|x - y\|.$$

1. By the properties of the norm, $||a|| \geq 0$ for all $a \in V$, so $d(x, y) = ||x - y|| \geq 0$.
2. Suppose that $d(x, y) = 0$. Then $||x - y|| = 0$ and by the properties of the norm this means that $x - y = 0$. Hence, $x = y$.
3. $d(x, y) = ||x - y|| = || - (y - x)|| = | - 1| \cdot ||y - x|| = ||y - x|| = d(y, x)$
4. $d(x, z) = ||x - z|| = ||x - y + y - z|| \leq ||x - y|| + ||y - z|| = d(x, y) + d(y, z)$

□

So by the above proposition, every norm on a vector space induces a metric. But not every metric induces a norm since not every metric satisfies the homogeneity property. For example consider the following metric on the reals:

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Then $2d(2, 0) = 2$ but $d(4, 0) = 1$ thus $d(2 \cdot 2, 0) \neq 2 \cdot d(2, 0)$.

Examples of Normed Linear Spaces:

1. $\mathbb{R}^n$ with the following norm is a normed linear space. Let $x = (x_1, x_2, \ldots x_n) \in \mathbb{R}^n$, then we define

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

This norm is also known as the ℓ^2 norm. The proof that $\mathbb{R}^n$ is a normed linear space with the above norm follows the same form as the proof shown below for the case $n = 2$.

Proposition 2. *The space $\mathbb{R}^2$ with the given norm is a normed linear space.*

Proof. Let $x, y \in \mathbb{R}^2$ and let $\alpha \in \mathbb{R}$.

1. $||x||_2 = (\sum_{i=1}^{\infty} |x_i|^2)^{1/2} = \sqrt{|x_1|^2 + |x_2|^2} = \sqrt{x_1^2 + x_2^2} \geq 0$ by definition of absolute value on $\mathbb{R}$ and the square root function.

2. $\|x\|_2 = \sqrt{x_1^2 + x_2^2} = 0$ which is true if and only if $x_1 = x_2 = 0$. So $\|x\|_2 = 0$ if and only if $x = 0$.

3. $\|\alpha x\|_2 = \sqrt{(\alpha x_1)^2 + (\alpha x_2)^2} = |\alpha|\sqrt{x_1^2 + x_2^2} = |\alpha|\|x|\|_2$.

4. $\|x+y\|_2^2 = \sum_{i=1}^{2} |x_i + y_i|^2 = |x_1 + y_1|^2 + |x_2 + y_2|^2$

$$\leq (|x_1| + |y_1|)^2 + (|x_2| + |y_2|)^2 = \left(|x_1|^2 + |x_2|^2\right) + \left(|y_1|^2 + |y_2|^2\right) + 2|x_1||y_1| + 2|x_2||y_2|$$

$$= \|x\|_2^2 + \|y\|_2^2 + 2\left(\sum_{i=1}^{2} |x_i y_i|\right)$$

By the Cauchy Schwartz Inequality (see Lemma below), $\sum_{i=1}^{2} |x_i y_i| \leq \|x\|_2 \; \|y\|_2$, thus,

$$\|x\|_2^2 + \|y\|_2^2 + 2\left(\sum_{i=1}^{2} |x_i y_i|\right) \leq \|x\|_2^2 + \|y\|_2^2 + 2\|x\|_2 \; \|y\|_2 = \left(\|x\|_2 + \|y\|_2\right)^2$$

Hence, $\|x+y\|_2 \leq \|x\|_2 + \|y\|_2$.

The Cauchy-Schwartz Inequality: In the normed linear space $\mathbb{R}^n$, where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, y_3, \ldots, y_n)$ and $\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$,

$$x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \leq \|x\| \; \|y\|.$$

Therefore, ($\mathbb{R}^2, \|\cdot\|$) is a normed linear space. □

2. Let $1 \leq p < \infty$, and define ℓ^p by $\ell^p := \{a = \{a_n\}_{n=1}^{\infty} : \sum_{n=1}^{\infty} |a_n|^p < \infty\}$.

We define the norm as the function $\|\cdot\|_p : \ell^p \to [0, \infty)$, given by $\|a\|_p = (\sum_{n=1}^{\infty} |a_n|^p)^{\frac{1}{p}}$

ℓ^p equipped with $\|\cdot\|_p$ is a normed linear space, and the elements of ℓ^p are said to be p-summable.

Proposition 3. *ℓ^p equipped with $\|\cdot\|_p$ is a normed linear space.*

The proof of this proposition follows the same steps as the proof for $\mathbb{R}^2$ with infinitely many coordinates in x and y, except instead of needing the Cauchy-Schwartz Inequality to prove the Triangle Inequality (step 4), we use Holder's Inequality and Minkowski's Inequality.

Holder's Inequality[1] For $p, q \in [1, +\infty)$, with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|ab\|_1 \leq \|a\|_p \|b\|_q.$$

Minkowski's Inequality[1] For $p \in [1, +\infty)$, if $\|a\|_p = \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{\frac{1}{p}} < \infty$ and $\|b\|_p = \left(\sum_{n=1}^{\infty} |b_n|^p\right)^{\frac{1}{p}} < \infty$, then

$$\|a+b\|_p \leq \|a\|_p + \|b\|_p.$$

3 Banach Spaces

We are now almost ready to define the central topic of this paper: Banach spaces. This section will do this and give some examples.

Definition[6]: A sequence in a normed linear space is called *Cauchy* if and only if for all real numbers $\epsilon > 0$ there exists an index $M > 0$ such that $\|a_m - a_n\| < \epsilon$ whenever $m, n \geq M$.

Definition[6]: A normed linear space $(V, \|\cdot\|)$ is called *complete* if and only if every Cauchy sequence in V converges to a point in V.

The definition of a Cauchy sequence rigorously formalizes the notion of a sequence in which any two terms of the sequence get arbitrarily close together as one moves further along in the sequence. This idea of a complete space is the final piece needed to define the primary focus of this paper.

Definition[6]: A *Banach space* is a complete normed linear space.

Examples of Banach Spaces

There are many Banach spaces that undergraduate mathematics majors work with but don't recognize as Banach spaces. For example, the set of real numbers, $\mathbb{R}$, with the norm $\|x\| = |x|$ as the distance from x to the origin is a Banach space. Also $\mathbb{R}^2$ with the usual distance function $\|x\|_2 = \sqrt{x_1^2 + x_2^2}$ is a Banach space. In fact we can generalize this in the following theorem, but first a lemma that will be useful in the proof of this theorem.

Lemma 1. *Let $1 \leq p \leq \infty$. Then for every $x \in \mathbb{R}^n$, $||x||_\infty \leq ||x||_p \leq n^{1/p}||x||_\infty$.*

Proof. Suppose $1 \leq p \leq \infty$. Let $x \in \mathbb{R}^n$. Then, for all i, $|x_i| = (|x_i|^p)^{1/p} \leq \left(\sum_{k=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$, so

$$\|x\|_\infty = \sup_i |x_i| \leq \left(\sum_{k=1}^{n} |x_i|^p\right)^{\frac{1}{p}} = \|x\|_p.$$

Also, $|x_i|^p \leq (\sup |x_i|)^p$, therefore $\sum_{k=1}^{n} |x_i|^p \leq n(\sup |x_i|)^p$. Raising both sides of the inequality to the $1/p$,

$$\|x\|_p = \left(\sum_{k=1}^{n} |x_i|^p\right)^{1/p} \leq n^{\frac{1}{p}} \sup |x_i| = n^{\frac{1}{p}} \|x\|_\infty.$$

Thus, for every $x \in \mathbb{R}^n$, $||x||_\infty \leq ||x||_p \leq n^{1/p}||x||_\infty$. □

Theorem 2 (6)**.** *$\mathbb{R}^n$ is a Banach Space with norm given by $|| \cdot ||_2 : \mathbb{R}^n \to \mathbb{R}$ defined by $||(x_1, x_2, \ldots, x_n)||_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ for all $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$.*

Proof. Let $||\cdot||_2 : \mathbb{R}^n \to \mathbb{R}$ be defined by $||(x_1, x_2, \ldots, x_n)||_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ for all $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$.

We first show that $||\cdot||_2$ is a norm.

1. $||\mathbf{x}||_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \geq 0$ by definition of square root.
2. $||\mathbf{x}||_2 = 0 \iff \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = 0 \iff x_1^2 + x_2^2 + \cdots + x_n^2 = 0 \iff x_i = 0$ for all $i \iff \mathbf{x} = \mathbf{0}$
3. $||\alpha\mathbf{x}||_2 = ||\alpha(x_1, x_2, \ldots, x_n)||_2 = ||(\alpha x_1, \alpha x_2, \ldots, \alpha x_n)||_2$

   $$= \sqrt{\alpha^2 x_1^2 + \alpha^2 x_2^2 + \cdots + \alpha^2 x_n^2} = \sqrt{\alpha^2(x_1^2 + x_2^2 + \cdots + x_n^2)}$$

   $$= \sqrt{\alpha^2}\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = |\alpha|||\mathbf{x}||_2$$
4. $\|x + y\|_2^2 = (x_1 + y_1)^2 + (x_2 + y_2)^2 + \cdots + (x_n + y_n)^2$

   $$= x_1^2 + 2x_1y_1 + y_1^2 + x_2^2 + 2x_2y_2 + y_2^2 + \cdots + x_n^2 + 2x_ny_n + y_n^2$$

   $$= (x_1^2 + +x_2^2 + \cdots x_n^2) + (y_1^2 + y_2^2 + \cdots + y_n^2) - 2(x_1y_1 + x_2y_2 + \cdots x_ny_n)$$

   $$= \|x\|_2^2 + \|y\|_2^2 - 2(x_1y_1 + x_2y_2 + \cdots x_ny_n)$$

   $$\leq \|x\|_2^2 + \|y\|_2^2 + 2|x_1y_1 + x_2y_2 + \cdots x_ny_n|$$

   $$\leq \|x\|_2^2 + \|y\|_2^2 + 2\|x\|_2 \, \|y\|_2$$

   $$= (\|x\|_2 + \|y\|_2)^2$$

   Therefore, $\|x + y\|_2 \leq \|x\|_2 + \|y\|_2$.

   So, we have proven that $\|\cdot\|$ is a norm.

Now we prove the space is complete. Let $\{X_i = (x_1, x_2, \ldots, x_n)_i\}_{i=1}^{\infty}$ be a Cauchy sequence in $\mathbb{R}^n$. By the preceding lemma, $|x_{j(i)} - x_{k(i)}| \leq ||X_j - X_k||$. Let $\epsilon > 0$ be given. There exists an index $M > 0$ such that for all $j, k \geq M$, $||X_j - X_k|| < \epsilon$. Hence, each sequence consisting of the jth coordinate of each entry in $\{X_i\}$ is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, each of these sequences converges to a real number. Let $X = (x_1, x_2, \ldots, x_n)$ be an arbitrary point in $\mathbb{R}^n$. Let $C = n^{1/p}$ if $1 \leq p < \infty$ and $C = 1$ if $p = \infty$. Again using the previous lemma,

$$||X_k - x|| \leq C \max\{|x_{k(i)} - x_i| : i = 1, 2, \ldots, n\}.$$

Let $\epsilon > 0$ be given. Choose and index $M_i \in \mathbb{N}$ such that $|x_{k(i)} - x_i| < \epsilon/C$ for all $k \geq N_i$, and let $N = max\{N_1, N_2, \ldots, N_n\}$. Then if $k > N$, $||X_k - X|| < \epsilon$, and thus the sequence converges to a point in $\mathbb{R}^n$. Hence, $\mathbb{R}^n$ is complete when given the norm $||x|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. □

Proposition 4. ℓ^∞ *with the norm* $||\mathbf{x}||_\infty = \sup_{\mathbf{i}} |\mathbf{x_i}|$ *is a Banach space.*

Proof. Let $X = \ell^\infty$, the set of bounded sequences of real numbers. Give X the following norm $||x||_\infty = \sup_i |x_i|$

1. This is indeed a norm:

   (a) Since the absolute value is always nonnegative, $\sup_i |x_i| \geq 0$, and since $|x| = 0$ if and only if $x = 0$, $\sup_i |x_i| = 0$ if and only if $x_i = 0$ for all i.

   (b) Let α be a fixed real number. By the properties of the absolute value $|\alpha x_i| = |\alpha||x_i|$ for all i. So

   $\sup_i |\alpha x_i| = \sup_i (|\alpha||x_i|)|\alpha| \sup_i |x_i|$. since α doesn't depend on i.

   (c) $||x + y||_\infty = \sup_i |x_i + y_i| \leq \sup_i (|x_i| + |y_i|)$ by the triangle inequality for real numbers.

   $$= \sup_i |x_i| + \sup_i |y_i|$$

   Hence $||\cdot||_\infty$ is a norm.

2. We now show ℓ^∞ is complete by showing any Cauchy sequence converges to an element of l^∞.

   Let $x^1, x^2, x^3, ...$ be a Cauchy sequence in l^∞.

   Note each element of $\{x^n\}$ is an element of l^∞, say $x^n = (x_1^n, x_2^n, x_3^n, \dots)$

   Let $\epsilon > 0$ be given. Then since the sequence is Cauchy, there exists an index M such that for all $n, m \geq M$, $||x^n - x^m||_\infty < \epsilon$.

   This means for all $n, m \geq M$, $\sup_i |x_i^n - x_i^m| < \epsilon/2$.

   It follows that for each k, the sequence $x_k^1, x_k^2, x_k^3, \dots$ is a Cauchy sequence in $\mathbb{R}$ which is a complete space. Thus, for each k, the sequence $x_k^1, x_k^2, x_k^3, \dots$ converges to an element $x_k \in \mathbb{R}$; i.e., $\lim x_k^n = x_k$.

   Set $x = (x_1, x_2, x_3, \dots)$.

   We now show $\{x^n\}$ converges to x:

   We know that for all k and all $m, n \geq M$, $|x_k^n - x_k^m| < \epsilon/2$.

   So, taking the limit as $m \to \infty$ we have $|x_k^n - x_k| \leq \epsilon/2$ for all k and all $n \geq M$.

Taking the supremum over all k gives us: $\sup_k |x_k^n - x_k| \leq \epsilon/2$ for all $n \geq M$.

This means $||x^n - x||_\infty \leq \epsilon/2 < \epsilon$ for all $n \geq M$, and thus, x^n converges to x.

If $x \in l^\infty$ then we are done. Each x^n is in ℓ^∞, so each x_j in each x^n is less than or equal to some number C_j. Therefore, since each x_n in the limit sequence is the limit of a sequence of x_j's, the entire sequence x is bounded by $\sup_j(C_j)$. Hence, $x \in \ell^\infty$ and so ℓ^∞ is a Banach space.

□

Example of a normed space that is not complete:

Not all normed linear spaces are complete; if they were there would be no need to give complete normed linear spaces a special name! We now give an example of a normed linear space that is not a Banach space.

Proposition 5. *Let $X = C([0,1])$, the set of all real valued continuous functions with domain [0, 1]. Give X the following norm*

$$||f||_1 = \int_0^1 |f(x)|\, dx.$$

The space $(C([0,1]), ||\cdot||_1)$ is a normed linear space that is not complete.

Proof. 1. $||f||_1 = \int_0^1 |f(x)|\, dx \geq 0$ because of positivity of the integral.

2. $||f||_1 = \int_0^1 |f(x)|\, dx > 0$ since f is continuous, unless $f(x) = 0$ for all $x \in [0,1]$ in which case $||f||_1 = \int_0^1 0\, dx = 0$
3. $||\alpha f(x)||_1 = \int_0^1 |\alpha f(x)|\, dx = |\alpha| \int_0^1 |f(x)|\, dx = |\alpha| ||f(x)||_1$.
4. by the Triangle Inequality for real numbers and the linearity of integrals;

$$||f_1 + f_2||_1 = \int_0^1 |f_1(x) + f_2(x)|\, dx \leq \int_0^1 |f_1(x)| + |f_2(x)|\, dx = \int_0^1 |f_1(x)|\, dx + \int_0^1 |f_2(x)|\, dx = ||f_1||_1 + ||f_2||_1$$

Now we show X is not complete by producing a Cauchy sequence that converges to a function not in $C([0,1])$.

For each $n = 1, 2, 3, \ldots,$ define $f_n(x) = x^n$.

Let $\epsilon > 0$ be given. The sequence $\{1/n\}_{n=1}^\infty$ converges to 0, so choose an index $M > 1/\epsilon$, then if $n \geq M$, $1/n < \epsilon$.

Then for all $n, m \geq M$, $||f_n - f_m||_1 = \displaystyle\int_0^1 |x^n - x^m|\, dx$. Without loss of generality, say $n < m$. Then on $[0,1]$, $x^m \leq x^n$ so that $\displaystyle\int_0^1 |x^n - x^m|\, dx = \int_0^1 x^n - x^m\, dx = \left[\frac{x^{n+1}}{n+1} - \frac{x^{m+1}}{m+1}\right]_0^1 = \frac{1}{n+1} - \frac{1}{m+1} < \frac{1}{n+1} < 1/n < \epsilon.$

Thus, the sequence is Cauchy.

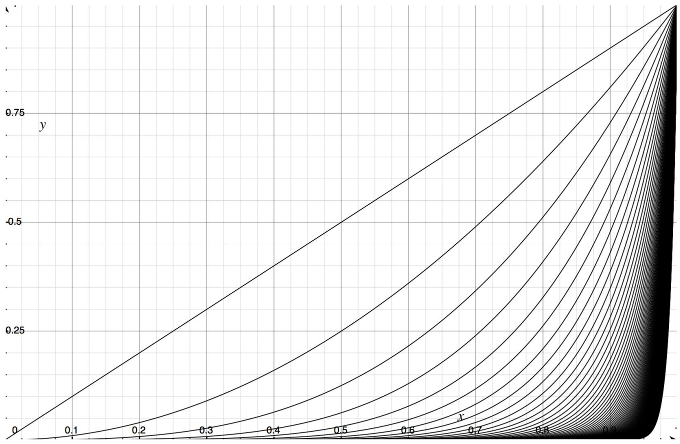

Figure 10: $f_n(x) = x^n$, $1 \le n \le 100$

We now show that the sequence converges to $f(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1 \end{cases}$

Clearly $f \notin C([0,1])$. And $f_n(x) - f(x) = \begin{cases} x^n & x \neq 1 \\ 0 & x = 1 \end{cases}$. This function is continuous except at one point, $x = 1$, so we can integrate it.

$||f_n - f||_1 = \int_0^1 |f_n(x) - f(x)|\, dx = \left[\frac{x^{n+1}}{n+1} \right]_0^1 = \frac{1}{n+1}$. Again, given an $\epsilon > 0$, we can find an $M > 0$ so that if $n \ge M$, $\frac{1}{n} < \epsilon$. Thus $f_n \to f$ and so this normed linear space is not complete since we have produced a Cauchy sequence of continuous functions that converges to a non-continuous function using the norm $|| \cdot ||_1$. □

We can make $C([0,1])$ into a Banach space by changing its norm.

Change the norm for $C([0,1])$ to $||f||_\infty = \sup_{x \in [0,1]} |f(x)| = \max_{x \in [0,1]} |f(x)|$ since the domain of the functions is a closed and bounded interval and the functions are continuous.

Proposition 6. *The space $(C([0,1]), || \cdot ||_\infty)$ is a Banach space.*

Proof. Let $X = C([0,1])$ with $||f||_\infty = \max_{x \in [0,1]} |f(x)|$.

(a) We first show $|| \cdot ||_\infty$ is a norm:

Clearly, $||f||_\infty \ge 0$ because $|f(x)| \ge 0$ for all $x \in [0,1]$.

Next, $||f||_\infty = 0$ if and only if $\max_{x \in [0,1]} |f(x)| = 0$ which means that $f(x) = 0$ for all $x \in [0,1]$.

Let $\alpha \in \mathbb{R}$, then $||\alpha f||_\infty = \max_{x \in [0,1]} |\alpha f(x)| = |\alpha| \max_{x \in [0,1]} |f(x)| = |\alpha| ||f||_\infty$.

Let f and $g \in C([0,1])$. For all $x \in [0,1]$, $|f(x)+g(x)| \leq |f(x)|+|g(x)|$. So that for all $x \in [0,1]$, $|f(x)+g(x)| \leq \max_{x\in[0,1]} |f(x)| + \max_{x\in[0,1]} |g(x)|$. Hence, $\max_{x\in[0,1]} |f(x)+g(x)| \leq \max_{x\in[0,1]} |f(x)| + \max_{x\in[0,1]} |g(x)|$. Thus, $||f+g||_\infty \leq ||f||_\infty + ||g||_\infty$, and $||\cdot||_\infty$ is a norm.

(b) Now show we have a complete space. Let $\{f_n\}$ be a Cauchy sequence in $C([0.1])$. Let $\epsilon > 0$ be given. There exists $M > 0$ such that for all $n, m \geq M$, $||f_n - f_m||_\infty < \dfrac{\epsilon}{3}$. This means $||f_n - f_m||_\infty = \max_{x\in[0,1]} |f_n(x) - f_m(x)| < \dfrac{\epsilon}{3}$ which implies that $|f_n(x) - f_m(x)| < \dfrac{\epsilon}{3}$ for all $x \in [0,1]$. Hence, for fixed $x \in [0,1], \{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$ which is complete. Thus for each $x \in [0,1]$ there exists an $\hat{x}$ such that $f_n(x)$ converges to $\hat{x}$. Set $f(x) = \hat{x}$ and we can see for some $M' > 0$ if $n \geq M'$ $|f_n(x) - f(x)| < \dfrac{\epsilon}{3}$. Also note that each f_n is continuous, so there is a $\delta > 0$ such that when $|x - x_0| < \delta, |f_n(x_0) - f_n(x)| < \frac{\epsilon}{3}$.

Show f is continuous at $a \in [0,1]$. For n large enough, and $|x-a| < \delta$, $|f(x) - f(a)| = |\hat{x} - \hat{a}| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$. Therefore, f is continuous on $[0,1]$.

Next we need to show that f_n converges to f; i.e. show $||f_n - f||_\infty = \max_{x\in[0,1]} |f_n(x) - f(x)|$ converges to 0 as $n \to \infty$. So, let $\epsilon > 0$ be given.

Then, $|f_n(x) - f(x)| = |f_n(x) - \lim_{m\to\infty} f_m(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)| \leq \lim_{m\to\infty} \max_{x\in[0,1]} |f_n(x) - f_m(x)| = \lim_{m\to\infty} ||f_n - f_m||_\infty \leq \dfrac{\epsilon}{3} < \epsilon$. Therefore, f_n converges to f, and the normed linear space is complete. □

Other well-known Banach Spaces are

1. The complex numbers $\mathbb{C}$ with norm given by the magnitude $||z|| = \sqrt{Re(z)^2 + Im(z)^2}$ is a Banach Space.

2. The space ℓ^p of p-summable sequences with norm given by $||x||_p = (\sum_{n=1}^{\infty} |x|^p)^{\frac{1}{p}}$ is a Banach space.

4 Finite Versus Infinite Dimensional Spaces

There are many differences between finite and infinite dimensional normed linear spaces. We will next look at three areas where these differences become apparent. In particular, we will examine:

1. the equivalence of certain normed linear spaces
2. the compactness of the closed unit ball
3. the continuity of linear functionals.

4.1 Equivalence of Finite Dimensional Normed Linear Spaces

If we are to discuss the “dimension” of a linear space, it would behoove us to give a rigorous definition of this term before proceeding.

Definition[10]: A linear space V is said to be *finite dimensional with dimension* n provided that there exists a subset B of V (called the *basis* for V) with $B = \{e_1, e_2, ..., e_n\}$ such that every element $x \in V$ can be written in the form $x = a_1e_1 + a_2e_2 + \cdots + a_ne_n$.

For example, $\mathbb{R}^3$ has $B = \{(1, 0, 0)), (0, 1, 0), (0, 0, 1)\}$ and for $(a, b, c) \in \mathbb{R}^3$ we can write $(a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1)$.

A linear space that is not finite dimensional is called *infinite dimensional* and has an infinite set $B = \{e_1, e_2, e_3, \dots\}$ such that every element $x \in V$ can be written in the form $x = a_1e_1 + a_2e_2 + a_3e_3 \cdots$.

For example, ℓ^2 has $B = \{e_1, e_2, e_3, \dots\}$ where each e_j is an infinite sequence with all zero's as the terms except for its jth term which is a one. So for $x = (1, \frac{1}{2}, \frac{1}{3}, \dots) \in \ell_2$ we can write $x = e_1 + \frac{1}{2}e_2 + \frac{1}{3}e^3 + \cdots$.

Definition[10]: A *linear map* is a function $f : X \to Y$, where X, Y are linear spaces, that has the following properties:

1. $f(a + b) = f(a) + f(b), \forall a, b \in X$
2. $sf(a) = f(sa) \forall a \in X$ and $s \in \mathbb{R}^3$.

A simple example of a linear map is the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$. By the properties of multiplication and addition of real numbers, if $x_1, x_2, a, b \in \mathbb{R}$,

$$f(ax_1 + bx_2) = ax_1 + bx_2 = af(x_1) + bf(x_2).$$

It is often useful to know which linear spaces are "essentially the same", meaning that statements about one space will hold true in another. To formalize this notion we use several related definitions.

Definition[5]: Two linear spaces X and Y are *linearly isomorphic* iff there exists a one-to-one and onto linear map $T : X \to Y$.

Definition[5]: If X and Y are normed linear spaces and T, T^{-1} are bounded (hence, continuous) linear maps, then X and Y are called *topologically isomorphic* or *homeomorphic.*

This brings us to the first major difference between finite and infinite dimensional space.

Theorem 3 (7)**.** *All finite dimensional normed linear spaces of dimension n are topologically isomorphic to $\mathbb{R}^n$.*

What this theorem tells us is that every finite dimensional space is a Banach space!

This theorem does not apply to infinite dimensional normed linear spaces; that is, there is no normed linear space which is topologically isomorphic to all infinite dimensional normed linear spaces.

4.2 Compactness and the Closed Unit Ball

As with linear spaces, we can define a notion for two norms being "essentially the same." To formalize this notion we first need the idea of "compactness."

Definition[10]: In a finite dimensional normed linear space V , a subset M of V is said to be *compact* if it is closed and bounded.

"Closed" meaning it contains all points on its boundary and "bounded" in the sense that it can be obtained in what we will call a ball. This brings us to the second difference between finite and infinite dimensional spaces.

Theorem 4 (6). *In a finite dimensional normed linear space* $(V, \|\cdot\|_n)$, *the closed unit ball* $\overline{B_n}(0,1) = \{x \in V : \|x\|_n \leq 1\}$ *is compact.*

Proof. Clearly the unit ball $\overline{B_n}(0,1) = \{x \in V : \|x\|_n \leq 1\}$ in $\mathbb{R}^n$ is closed and bounded. Since all n dimensional normed linear spaces are topologically isomorphic to $\mathbb{R}^n$, and all topological concepts such as closed and bounded carry over to homeomorphic spaces, the closed unit ball in any finite dimensional space is compact.

□

In infinite dimensional space this is not true as we can see in this next example. First we need a more generalized definition of compactness that includes the one we gave for finite spaces as well as extends to infinite dimensions.

Definition[6]: In any normed linear space, V, a set is *compact* provided every sequence in V has a convergent subsequence.

Consider ℓ^2 with the 2-norm $\|x\|_2 = \sqrt{x_1^2 + x_2^2 + x_3^2 + \cdots}$. It is an infinite dimensional space with the basis set $Y = \{e_j : j = 1, 2, 3, ...\}$ where e_j is the infinite sequence with all zeroes except in the jth coordinate it has a one. This means every element $a = (a_1, a_2, a_3, \dots)$ in ℓ_2 can be written in the form

$$a = a_1 e_1 + a_2 e_2 + a_3 e_3 + \cdots .$$

Let $\overline{B_2}(0,1)$ be the closed unit ball in ℓ^2. Then the set Y is a sequence in the closed unit ball but it has no convergent subsequence since $\|e_j - e - k\|_2 = \sqrt{2}$ for all $j \neq k$

4.3 The Continuity of Linear Functionals

We start the section on the third difference between finite and infinite dimensions with the definition of the namesake of Funcitonal Analysis.

Definition[6]: Let V be a vector space with scalars in a field $\mathbb{K}$. Then a linear map $f : V \to \mathbb{K}$ is called a *linear functional.*

Examples of Linear Functionals

1. Let $\mathfrak{F}([0,1])$ be the space of all real-valued functions of a real variable that are integrable on the closed interval $[0,1]$. Then the function $F : \mathfrak{F}([0,1]) \to \mathbb{R}$ defined by

$$F[f] = \int_0^1 f(x)dx$$

for all $f \in \mathfrak{F}([0,1])$, is a linear functional. We show this here:

By the properties of integrals,

$$F[\alpha f + \beta g] = \int_0^1 [\alpha f(x) + \beta g(x)]dx$$

$$= \int_0^1 \alpha f(x)dx + \int_0^1 \beta g(x)dx$$

$$= \alpha \int_0^1 f(x)dx + \beta \int_0^1 g(x)dx$$

$$= \alpha F[f] + \beta F[g].$$

So f is indeed a linear map.

2. Let $\mathbb{R}[X]$ be the space of all polynomials with real coefficients, and fix $\alpha \in \mathbb{R}$. Then the function $\pi_\alpha : \mathbb{R}[X] \to \mathbb{R}$ defined by

$$\pi_\alpha[p] = p(\alpha)$$

for all $p \in \mathbb{R}[X]$, is a linear functional.

3. Let $M(2,\mathbb{R})$ be a the space of all 2×2 matrices with real entries. Then the function $det : M(2,\mathbb{R}) \to \mathbb{R}$ defined by

$$A \mapsto det(A)$$

for all $A \in M(2,\mathbb{R})$, is a linear functional.

We now come to the third difference between finite and infinite dimensions.

Theorem 5 (6). *If V is a finite dimensional normed linear space, then every linear functional $F : V \to \mathbb{R}$ is continuous.*

Proof. Let V be a finite dimensional normed linear space with norm $\|\cdot\|$. Since all n dimensional spaces are homeomorphic to $\mathbb{R}^n$, and all norms on $\mathbb{R}^n$ are equivalent, we can let $V = \mathbb{R}^n$ with the norm $\|x\| = \sum_{i=1}^n |x_i|$. Let $f : V \to \mathbb{R}$ be a linear functional. Let $B = \{b_1, b_2, \ldots, b_n\}$ be a basis for V, and let $B^* = \{e_1, e_2, \ldots, e_n\}$ be the standard basis for $\mathbb{R}$ where e_j is the n-tuple with 0's in each coordinate except for the jth coordinate which is equal to 1.

Let $\{x_k\}_{k=1}^{\infty} \subset V$ such that x_k converges to $x \in V$. We note that each $x_k = (x_{k_1}, x_{k_2}, \ldots, x_{k_n}) = \sum_{i=1}^{n} x_{k_i} b_i$, and $x = (x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i b_i$. Also note that $f(b_i) = e_i$ for all i.
Our functional is linear so

$$|f(x_k) - f(x)| = |f\left(\sum_{i=1}^{n} x_{k_i} b_i\right) - f\left(\sum_{i=1}^{n} x_i b_i\right)| = |\sum_{i=1}^{n} f\left(x_{k_i} b_i\right) - \sum_{i=1}^{n} f\left(x_i b_i\right)| = |\sum_{i=1}^{n} (x_{k_i}) f\left(b_i\right) - \sum_{i=1}^{n} (x_i) f\left(b_i\right))| =$$
$$|\sum_{i=1}^{n} (x_{k_i} - x_i) f(b_i)| \leq \sum_{i=1}^{n} |(x_{k_i} - x_i)|\, |f(b_i)| \leq M \sum_{i=1}^{n} |(x_{k_i} - x_i)| \text{ for } M = \max_i\{f(b_i)\}.$$

And $\sum_{i=1}^{n} |(x_{k_i} - x_i) = \|x_k - x\|$ is converging to 0 since $x_k \to x$ in V. Thus $f(x_k) \to f(x)$ in $\mathbb{R}$ and f is continuous.

□

This is not true for infinite dimensional spaces. For example, consider the normed linear space $C[0,1])$ of all continuous functions on $[0,1]$ with the sup norm $\|p\|_\infty = \sup_{x\in[0,1]} |f(x)|$. Let $f_n(x) = \dfrac{\sin(n^2 x)}{n}$, then $\|f_n\|_\infty$ converges to the zero function, $f(x) = 0$.

Now let $D : C([0,1]) \to \mathbb{R}$ be defined by $D(f) = f'(0)$, i.e., D is the derivative of the function evaluated at $x = 0$. D is a linear functional. We claim this linear functional is not continuous: Note $D(f_n) = \dfrac{n^2 cos(n^2 \cdot 0)}{n} = n$, so $D(f_n)$ is unbounded instead of $D(f_n)$ converging to $D(f) = 0$.

5 Hahn-Banach Theorems

Functionals on normed linear spaces are the central topic of study in Functional Analysis. As such, it is useful for Functional analysts that they have a way to "find" functionals which they can study. Two important theorems, called the "Hahn-Banach Theorems" allow mathematicians to construct these functionals. In this section we will give the necessary background for understanding the Hahn-Banach theorems, culminating in their statements. We start with a definition and a theorem.

Definition[8]: The set of all linear maps from a vector space V to its underlying scalar field $\mathbb{K}$ is called the *algebraic dual space of* V and is denoted $V^{'}$.

Theorem 6 (8). *$V^{'}$ is a vector space.*

Proof. Let $V = V(\mathbb{K})$ be a vector space over a field $\mathbb{K}$ and $V^{'}$ be the set of all linear maps $T : V \to \mathbb{K}$. In order to prove that this set is a vector space we must show that it satisfies the vector space axioms. Well, if T_1 and T_2 are linear maps from V to $\mathbb{K}$ then their sum $T_1 + T_2$ is also a linear map from V to K and thus is contained in $V^{'}$. The map T_0 that maps every input to zero is the additive identity because $T + T_0 = T_0 + T = 0 + T = T + 0 = T$. For every linear map T there is a map $-T = -1(T)$ such that $T + (-T) = -T + T = \mathbf{0}$. For every $s \in \mathbb{K}$ $sT(x) = T(sx) \in V^{'}$. The multiplicative identity is the identity of the field. For any $s, t \in \mathbb{K}, (s+t)T = sT + tT$ and $s(tT) = t(sT) = (st)T = stT$. Finally, $0T = 0$ because 0 multiplied by anything is 0. □

If we want to look at this collection of functionals as a Banach Space, we need to add continuity to the conditions on the linear functionals and a norm on the space. Again we remind the reader that we are considering the reals to be the field $\mathbb{K}$.

Definition[8]: The set of all continuous linear maps from a vector space V to its underlying scalar field $\mathbb{R}$ is called the *topological dual space of* V and is denoted V^*.

We note that the topological dual's notation V^* comes from the fact that all continuous linear functions are bounded (and vice versa), meaning that for $f : V \to \mathbb{R}$ and $x \in V$, $\|f(x)\| \leq M$ for some $M \in \mathbb{R}$.

We give V^* the norm $\|F\| = \sup\limits_{x \in V, x \neq 0} \dfrac{|F(x)|}{\|x\|}$. This norm is equivalent to $\sup\limits_{\|x\|=1} |F(x)|$ because $\frac{|F(x)|}{||x||} = |\frac{1}{||x||} F(x)| = |F(\frac{x}{||x||})|$ and $\frac{x}{||x||}$ is a unit vector.

Proposition 7. *V^* with the above norm is a Banach space.*

Proof. Consider V^* and we define for $F \in V^*$, $\|F\| = \sup\limits_{\|x\|=1} |F(x)|$. We first show this is a norm by considering the four properties of the norm.

(a) By definition $\|F\| \geq 0$.

(b) $\|F\| = 0$ if and only if $|F(x)| = 0$ for all $x \in V$ if and only if F is the zero function.

(c) Let $\alpha \in \mathbb{R}$. Then $\|\alpha F\| = \sup\limits_{\|x\|=1} |\alpha F(x)| = |\alpha| \sup\limits_{\|x\|=1} |f(x)| = |\alpha| \cdot \|F\|$.

(d) Let $F, G \in V^*$. Then $\|F + G\| = \sup\limits_{\|x\|=1} |F(x) + G(x)| \leq \sup\limits_{\|x\|=1} |F(x)| + \sup\limits_{\|x\|=1} |G(x)| \leq \|F\| + \|G\|$.

So we have a normed linear space. Next show that the space is complete.

Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in V^*. Note that $\forall x \in V$, $\{f_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$, since

$$\|f_n(x) - f_m(x)\| \leq \|f_n - f_m\||x|.$$

So we can define an $f : V \to \mathbb{R}$ by $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in V$. f is linear:

$$f(x+y) = \lim_{n\to\infty}(f_n(x+y)) = \lim_{n\to\infty}(f_n(x) + f_n(y)) = \lim_{n\to\infty} f_n(x) + \lim_{n\to\infty} f_n(y) = f(x) + f(y), \text{ and}$$

$$f(\alpha x) = \lim_{n\to\infty}(f_n(\alpha x)) = \lim_{n\to\infty}(\alpha f_n(x)) = \alpha \lim_{n\to\infty} f_n(x) = \alpha f(x).$$

Also, f is bounded: $|f(x)| = |\lim_{n\to\infty} f_n(x)| = \lim_{n\to\infty} |f_n(x)| \leq (\sup\|f_n\|)|x| \leq K|x|$ for some $K \in \mathbb{R}$.

Next we must show that $\|f_n - f\| \to 0$.

$$|f_n(x) - f(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)| \leq \limsup_{m\to\infty} \|f_n - f_m\|$$

and since x is arbitrary we obtain,

$$\|f_n - f\| \leq \limsup_{m\to\infty} \|f_n - f_m\|.$$

The quantity on the right can be made very small because the sequence $\{f_n\}$ is Cauchy, and we are done. [8]

□

Functional analysts are interested in whether the dual is "large enough" to answer these questions:

extension: Suppose that $Z \subseteq V$ is a subspace of V and $f \in Z^*$. Can we construct a linear functional $\hat{f} \in V^*$ such that $f = \hat{f}$ on Z?

point separation: for $x \neq y \in V$, can we find $f \in V^*$ such that $f(x) \neq f(y)$?

This leads us to the two Hahn-Banach theorems mentioned in the introduction to this section. These theorems are possibly the most important results of Functional Analysis. They allow Functional analysts to answer the extension and point separation questions by guaranteeing the existence of functionals such that the dual is "large enough."

Theorem 7. ***Hahn-Banach Theorem (analytic form).*** *[8] Let V be a real normed linear space and let M be a subspace of V. Let $f \in M^*$. Then there is a linear functional $\widetilde{f} \in V^*$ that extends f (that is, $f(x) = \widetilde{f}(x)$ for all $x \in M$) and $\|f\|_{M^*} = \|f\|_{V^*}$.*

This answers the extension question in the affirmative; in fact, even better, the theorem guarantees that the norms of the original functional and its extension will be equal! But what of the point separation question? Can we find a functional that "separates" two points? The answer to this question is given by the second form of the Hahn-Banach theorem. But first, let us rigorously define what it means for a functional to "separate" two points.

Definition[8]: Let V be a normed linear space. We say the set $H(F, c) = \{x \in V | F(x) = c\}$, where $F : V \to \mathbb{R}$ is a linear functional, *separates* A and B provided $F(a) > c$ for all $a \in A$ and $F(b) < c$ for all $b \in B$.

Theorem 8. ***Hahn-Banach Theorem (geometric form).*** *[8] Let A and B be non-empty convex subsets of a real normed linear space V such that $A \cap B = \emptyset$ and the interior of A is non-empty. Then there is a hyperplane that* *separates A and B; i.e., there exists a real number c and a continuous linear functional $F : V \to \mathbb{R}$ such that the set $H(F, c) = \{x \in V | F(x) = c\}$ separates A and B*

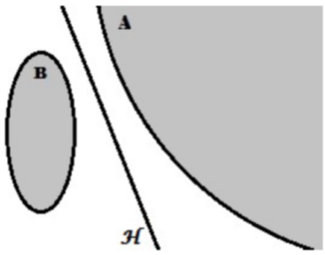

Figure 11: Hahn-Banach Theorem (geometric form) [8]

6 Consequences of the Hahn-Banach Theorems

As was previously stated, the Hahn-Banach Theorems are two of the most important theorems in Functional analysis. They are used in the proofs of several other theorems which describe the properties of functionals on Banach spaces. In this section we will discuss three such theorems; the Open Mapping Theorem, the Closed Graph Theorem, and the Bounded Inverse Theorem. We will then briefly give examples of other areas of mathematics where the Hahn-Banach Theorems see use.

We begin with some necessary definitions.

Definition[6]: Let $T : X \to Y$ be a linear operator between two normed linear spaces. Then we say that T is *bounded* if and only if there exists a number $M > 0$ such that $||T(x)||_Y \leq M||x||_X$ for all $x \in X$.

Definition: Let $f : X \to Y$ be a function. Then f is said to be *onto* if and only if for all $y \in Y$, there exists an $x \in X$ such that $f(x) = y$.

Definition[6]: Let $f : X \to Y$ be a function. Then f is said to be an *open map* if f maps open sets in the domain to open sets in the image.

This brings us to our first theorem.

Theorem 9. *The Open Mapping Theorem.[2] Let X and Y be Banach spaces, and let $F : X \to Y$ be a bounded, onto linear operator. Then F is an open map.*

The next theorem relates the continuity of a linear operator between Banach spaces with the topological properties of its graph.

Theorem 10. *The Closed Graph Theorem. [2] Let X and Y be Banach spaces, and let $F : X \to Y$ be a linear operator. Then F is continuous if and only if the graph of F, $Graph(F) = \{(x, F(x)) : x \in X\}$, is closed in $X \times Y$.*

Note that in this previous theorem, F is only required to be a linear operator.

Definition: Let $f : X \to Y$ be a function. Then f is said to be *bijective* if and only if it is onto and for all $a, b \in X$, if $f(a) = f(b)$, then $a = b$.

An important property of bijective functions is that they have inverses; that is if $f : X \to Y$ is a bijective function then there exists a bijective function $f^{-1} : Y \to X$ such that for all $x \in X$ and $f(x) \in Y$, $f^{-1}(f(x)) = x$. This brings us to the third and final theorem.

Theorem 11. *The Bounded Inverse Theorem [8]. Let X and Y be Banach spaces, and let $F : X \to Y$ be a bijective bounded linear operator. Then its inverse F^{-1} is continuous and bounded.*

We end the paper with a description of some of the uses of the Hahn-Banach Theorem. The theorems extended the study of the dual space and reached into other areas of mathematics such as [7]:

- Complex Analysis - Cauchy Integral Theorem for vector-valued analytic functions defined on a Banach space
- Partial Differential Equations -to solve the problem involving the existence of Green's function for a given boundary value problem for the Laplace operator.
- Game Theory- Minimax Theorem [3], which gives the optimal strategy for a game under certain circumstances.
- Applications to convex programming, which concerns the properties of convex sets

Clearly there is a lot to learn about Banach spaces. This paper only presents a small sampling of their properties and nuances. I hope to eventually expand my knowledge of Banach spaces further, possibly to the point where I am able to contribute new results to their study.

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