SPHERICAL TRIGONOMETRY

"THERE IS NO ROYAL ROAD TO GEOMETRY"

EUCLID

BY

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2016

Abstract

The purpose of this paper is to derive various trigonometric formulas for spherical triangles. The subject of spherical trigonometry has many navigational and astronomical applications.

History

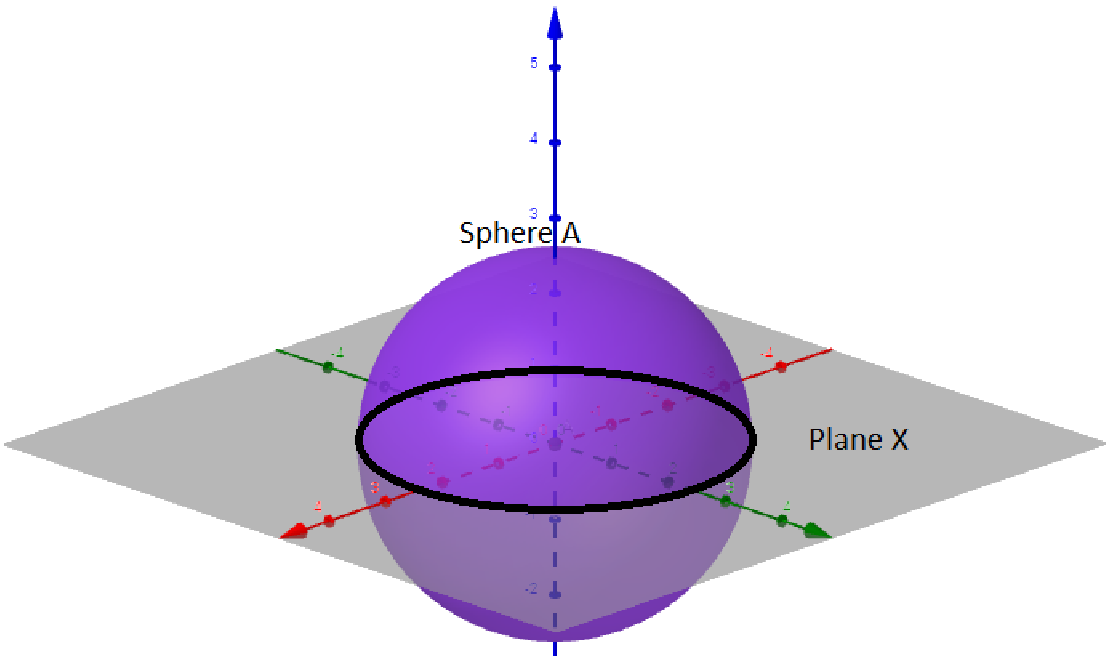
Geometry has been developing and evolving for many centuries. Its uses are vast and continue to affect our every day lives. The study of the sphere in particular has its own unique story, and has two major turning points. This study first began with the push of astronomy and was developed in depth by the Greeks. There is speculation that mathematical discoveries about the sphere were made as early as the second century, but there is no proof for this.[1] The major transition for the understanding of spherical geometry was the work of Menelaus of Alexandra. In his work *Spherica* the author delves deeply into the properties of the sphere and the calculations related to lengths and measures of a sphere's arcs. For a long time the equations he discovered, such as the measure of circumference of a sphere and the measure of arc lengths, were accepted and no further study was needed.

The next major motivation for learning spherical trigonometry was religious matters; the religion of Islam requires that the direction of Mecca is always known for daily prayer. Menelaus' findings were further developed during the Islamic Enlightenment period. There is some debate as to the discovery of the Law of Sines for spherical triangles. Possible sources of this discovery stem from the debate over the two Muslim scientists, Abū Nasr or Abū Í-Wafā'. Who ever was responsible for the progress in the Law of Sines allowed for a more concise proof to be developed later; as well as leading to other theorems and identities on spherical trigonometry. Another major name in geometry is Euclid. He made a big impact later in the third century;

though the system of spherical trigonometry does not incorporate parallel lines, Euclidean geometry gave some insight to spherical behavior. In his work *Elements* Euclid published equations which help lead us to the Pythagorean Theorem and the Law of Cosines. Though mathematicians brought insight to this area of study, many influences on spherical trigonometry also came from the field of science. Further discovery about the behavior of arcs and angles became prominent in the late Renaissance period. John Napier, a Scottish scientist who lived around the 17th century, was the first to work with right spherical triangles and the basic identities of these shapes. Using Napier's Rules, the law of cosines for spheres was discovered.[1]

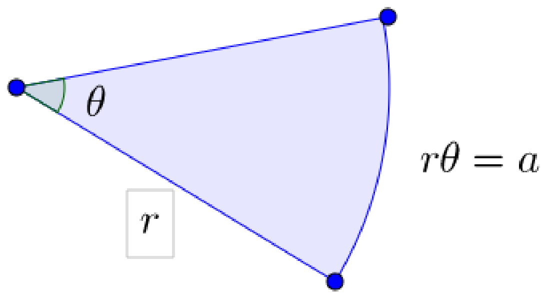
Definition 0.0.1. On a sphere, a **great circle** is the intersection of the sphere with a plane passing though the center, or origin, of the sphere.[1]

Example 0.0.2. Great Circle



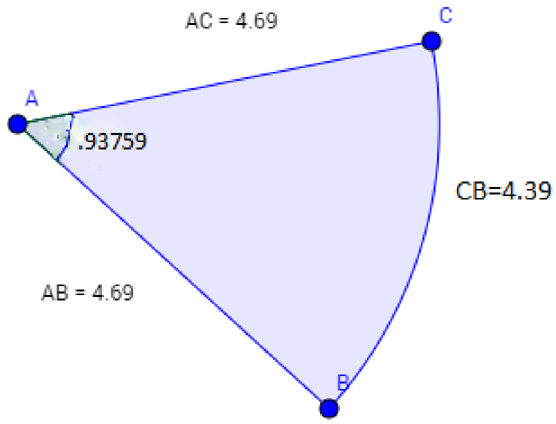
The solid line illustrates a great circle which is the intersection of the plane X and the sphere A. The plane in the 0.0.2 is the horizontal plane; however, the plane can have any orientation that bisects the sphere. The circle created by this intersection will have radius equal to the length of the radius of the sphere. It also follows that the length of the circumference of the great circle will equal $2\pi r$ where r is the length of the radius of the sphere. This is relevant because it enables us to calculate the length of a circular segment by considering the relation between the inner angle and the radius of the sphere. Therefore, for any sphere and any angle, the length of one arc segment will just equal $r\theta$ where θ is the measure of the angle in radians. This is true because each circular segment is just fraction of the entire circumference. We can show this relation of the angle and arc length in the example below.

Example 0.0.3. General arc measure (where a is the length of the arc)



This is a general example, but we can apply specific values as in the example on the following page.

Example 0.0.4. Measure of Arc



In this example we cut our a segment of great circle by relating an inner angle. For this example let $R = 4.69$ and $\theta = .93759$ radians, thus $(4.69)(.93759) = 4.39$, which is the measure of the length of the arc. This example was created in a dynamic mathematics software program which gave these related measurements.

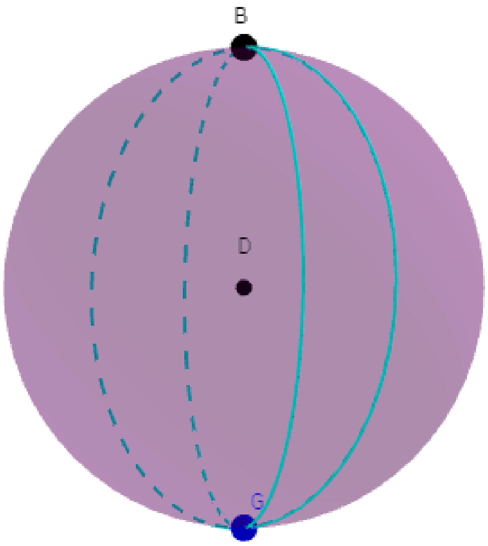
The next definition required in understanding a spherical triangle is that of a lune.

Definition 0.0.5. Lune

A lune is a part of the sphere which is captured between two great circles.[1]

This definition is relevant because it started the ability to capture shapes on a sphere. This definition itself is not extremely significant, but it is through this shape which we can form other shapes on the sphere. The shape of a lune can also be seen in the next figure.

Example 0.0.6. Lune



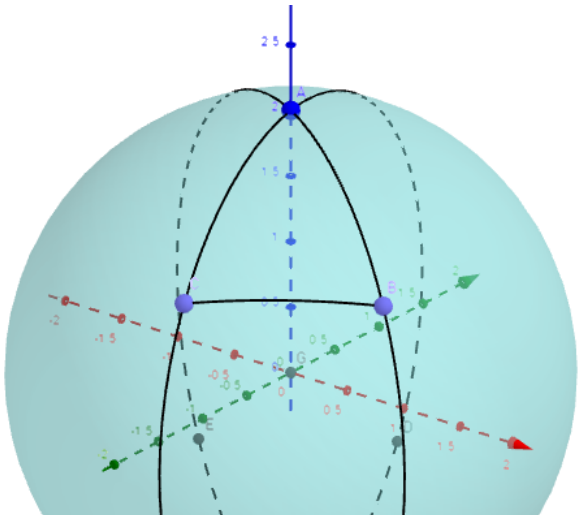
Spherical triangles can be defined in terms of lunes.

Definition 0.0.7. Spherical Triangle

A **spherical triangle** is the intersection of three distinct lunes.[1]

In the figure above we can consider that there are two lunes which are the on opposite sides of the sphere, it is natural that another lune bisecting these two will be needed. A simpler way to think about a spherical triangle is the shape captured through the intersection of three great circles. It is also interesting to consider that any two unique points, which are not diametric on the spheres surface, lie on a great circle. This makes sense because a triangle would have three vertices and therefore three different great circles which go through two different points. Since the spherical triangle's edges are curved, it is clear that the equations, sides, area, and general properties will be different from that of a planar triangle. We also know that in some way we should be able to relate the functions that we do discover to the radius of the sphere, because it seems natural that the formulas would change in relation to the sphere's radius.We know that the behavior of a spherical triangle will be very different from the planar triangles, there are even some definitions which can illustrate this for us.

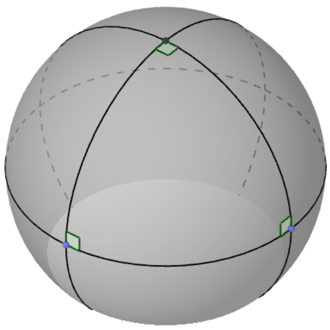
Example 0.0.8. Spherical Triangle



Definition 0.0.9. Spherical Excess is the amount by which the sum of the angles (in the spherical plane only) exceed $180°$.

This definition tells us about the behavior of the sphere and its edges. We know that the length of the edges on a spherical triangle will be greater the edges on a corresponding planar triangle, since they are curved. This definition allows for a spherical triangle to have multiple right angles.

Example 0.0.10. *Spherical triangle with three right angles*

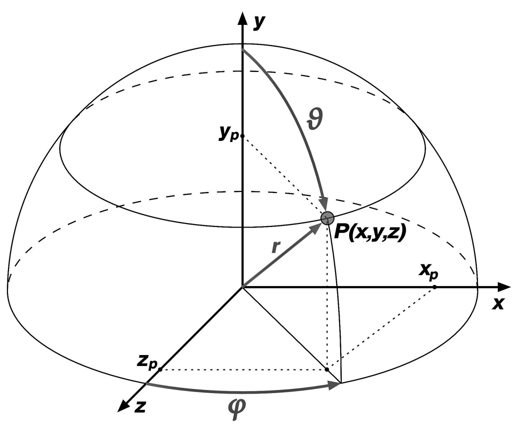


Definition 0.0.11. [3]

Spherical Coordinates is a coordinate system in three dimentions. The coordinate values stated below require r to be the length of the radius to the point P on the sphere. The value φ the angle between the z-axis, and the vector from the origin to point P, and θ the angle between the x-axis, and the same vector as in the figure 0.0.12. Then we can say the x, y, z coordinates are defined:

$$(r\ cos(\theta)sin(\varphi), r\sin(\theta)sin(\varphi), r\ cos(\varphi))$$

Example 0.0.12. *Spherical Coordinates in the xyz plane*



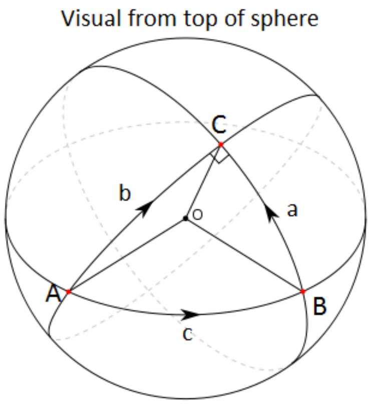
Theorem 0.0.13. ***The Spherical Pythagorean Theorem[3]***

For a right triangle, ABC on a sphere of radius r, with right angle at vertex C and sides length a, b, c is defined:

$$cos\left(\frac{c}{r}\right) = cos\left(\frac{a}{r}\right) \cdot cos\left(\frac{b}{r}\right)$$

This equation can equivalently be written as $cos(c) = cos(a)cos(b)$ when the radius is 1.

Proof. Consider a spherical triangle on the surface of a sphere with radius r, and sides a, b, c. Let vertex C in the spherical triangle be a right angle, as in the diagram below.[3]



Without loss of generality, suppose C is defined by spherical coordinates $C = (0, 0, r)$. Now consider the other vertices on the spherical triangle. Call one of these A and let it be defined using spherical coordinates defined above $A = (r\ sin\left(\frac{b}{r}\right), 0, r\ cos\left(\frac{b}{r}\right))$ where b is equal to the measure of the arc length along the great circle between A and C. Note $\frac{b}{r}$ is necessarily also equal the interior angle between the vector which goes from the O to C and the vector from O to A, since the radius is equal to r. Let the other vertex called B be defined $B = (0, r\ sin\left(\frac{a}{r}\right), r\ cos\left(\frac{a}{r}\right))$ where a is the arc length between B to C, by the same reasoning. Thus $\vec{A} = r\ sin\left(\frac{b}{r}\right)\hat{x} + r\ cos\left(\frac{b}{r}\right)\hat{z}$ and $\vec{B} = r\ sin\left(\frac{a}{r}\right)\hat{y} + r\ cos\left(\frac{a}{r}\right)\hat{z}$. Now consider $\vec{A} \cdot \vec{B}$

$$\begin{aligned}\vec{A} \cdot \vec{B} &= (r\ sin\left(\frac{b}{r}\right)\hat{x} + r\ cos\left(\frac{b}{r}\right)\hat{z} \cdot r\ sin\left(\frac{a}{r}\right)\hat{y} + r\ cos\left(\frac{a}{r}\right)\hat{z} \\ &= r^2 \cos\left(\frac{b}{r}\right) cos\left(\frac{a}{r}\right)\end{aligned}$$

Thereforesince the dot product of two vectors results in a scalar then $\vec{A} \cdot \vec{B} = r^2 \cos\left(\frac{b}{r}\right) cos\left(\frac{a}{r}\right)$. Now consider the magnitude of $|\vec{A} \cdot \vec{B}|$. Consider

$$\begin{aligned}|\vec{A} \cdot \vec{B}| &= ||A||B||cos\left(\frac{c}{r}\right) \\ |\vec{A}| &= r && \text{since the radius is } r \\ |\vec{B}| &= r && \text{since the radius is } r \\ |\vec{A} \cdot \vec{B}| &= r^2\ cos\left(\frac{c}{r}\right)\end{aligned}$$

Recall that the cosine of the angle between two vectors is just equal to the dot product of each vector, over the magnitude of each vector. We know that the sign will not matter since the magnitude takes the absolute value so the signs will be equal, now using this information in relation to the equations above we can conclude that

$$cos\left(\frac{c}{r}\right) = cos\left(\frac{a}{r}\right) cos\left(\frac{b}{r}\right)$$

$\square$

Theorem 0.0.14. ***Law of Cosines***

The Law of Cosines states that if a,b,c are the sides and A,B,C are the angles of a spherical triangle, then[1]

$$cos\left(\frac{c}{r}\right) = cos\left(\frac{a}{r}\right) cos\left(\frac{b}{r}\right) + sin\left(\frac{a}{r}\right) sin\left(\frac{b}{r}\right) cos(C)$$

Proof. Let there be a spherical triangle with sides denoted a, b, c. Let their opposing angles be labeled A, B, C, where $A \neq 90°$, $B \neq 90°$, and $C \neq 90°$. Let us call the center of the sphere to be S. Therefore we know from any single vertex, A, B, C, to the point S is exactly equal to the radius, r. If we arbitrary let A be the largest angle(since we can arbitrarily name any angle) then we can that there is a point somewhere between the vertices B and C which creates a right angle, since A is the largest angle and we must have spherical excess, it follows that creating a right angle is possible. Call this vertex N. This partition will create two right spherical triangles. Now we know that N some distance n from B and is the distance $(a - n)$ from vertex C to N. There will also be some distance we shall call y from the vertex N to point A. With this information we can use the spherical Pythagorean equation. So now applying the equation we are left with two equations

$$cos(b) = cos(n)cos(y) \qquad \text{Equation I}$$

$$cos(c) = cos(a - n)cos(y) \qquad \text{Equation II}$$

Now we can solve for $cos(y)$ in equation I so therefore we have:

$$cos(y) = \frac{cos(c)}{cos(a-n)}$$

Now we need to solve using the same process for equation II so we have:

$$\begin{aligned}
cos(y) &= \frac{cos(b)}{cos(n)} \\
\frac{cos(c)}{cos(a-n)} &= \frac{cos(b)}{cos(n)} && \text{combining equations} \\
cos(c)cos(n) &= cos(b)cos(a-n) && \text{cross multiply} \\
cos(c)cos(n) &= cos(b)(cos(a)cos(n) + sin(a)sin(n)) \\
cos(c)cos(n) &= cos(b)cos(a)cos(n) + cos(b)sin(a)sin(n)) \\
cos(c) &= cos(b)cos(a) + sin(a)sin(b)\frac{sin(n)}{cos(n)} && \text{divide by } cos(n) \\
cos(c) &= cos(a)cos(b) + sin(a)cos(b)tan(n) && \text{Napier}^{1}[1] \\
cos(c) &= cos(a)cos(b) + sin(a)sin(b)cos(C)
\end{aligned}$$

which gives the law of cosines as desired.

□

Also we can put this in terms of the radius of the sphere since we know that $\theta = \frac{x}{R}$ where x is the arc length and R is the radius of the sphere so if we substitute for the sides and angles with respect to the radius we have

[1] Napier discovered the substitution for this proof that $cos(b)tan(n) = sin(b)cos(C)$

$$cos\left(\frac{c}{r}\right) = cos\left(\frac{a}{r}\right) cos\left(\frac{b}{r}\right) + sin\left(\frac{a}{r}\right) sin\left(\frac{b}{r}\right) cos(C)$$

By equivalence we know that

$$\cos\left(\frac{a}{r}\right) = \cos\left(\frac{c}{r}\right)\cos\left(\frac{b}{r}\right) + \sin\left(\frac{c}{r}\right)\sin\left(\frac{b}{r}\right)\cos(A)$$

$$\cos\left(\frac{b}{r}\right) = \cos\left(\frac{a}{r}\right)\cos\left(\frac{c}{r}\right) + \sin\left(\frac{a}{r}\right)\sin\left(\frac{b}{r}\right)\cos(B)$$

This substitution can be made in any equation since the relation of inner angles to the arc length and r will always be consistent for that particular sphere.

Example 0.0.15. Flight Routes

Spherical trigonometry is very applicable to our every day life. One of these applications is discovering the length and directions needed for navigation. The formulas easily apply to the earth because the latitude and longitude lines are actually examples of great circles. The law of cosines can be used on the example below, in which we want to find the distance between two cities.

Given:

Earth's radius: 6,371 km

Distance from San Francisco, California to Dublin, Ireland: 8198 km

Distance from San Francisco, California to Seattle, Washington: 1094 km

Angle between these two arcs: 0.4763342 (radians)

Now let us say we wish to find the distance from Dublin to Seattle. To find this we use the equation below and then substitute the numerical values.

$$c = r \ \arccos\left(\cos\left(\frac{a}{r}\right)\cos\left(\frac{b}{r}\right) + \sin\left(\frac{a}{r}\right)\sin\left(\frac{b}{r}\right)\cos(C)\right)$$

$$c = 6371 \arccos\left(\cos\left(\frac{1094}{6371}\right)\cos\left(\frac{8850}{6371}\right) + \sin\left(\frac{1094}{6371}\right)\sin\left(\frac{8850}{6371}\right)\cos(.47633)\right)$$

$$c = 7232 \text{ km}$$

This value is an estimation because it assumes the earth is a perfect sphere with no dips or irregularities which is not true. This estimation is useful, however; since it was so quick and easy to find. Now we at least have some idea of the distance between two places that would otherwise be impossible to directly measure.

Theorem 0.0.16. ***Law of Sines[4]***

Where a,b,c are the sides of a spherical triangle and A,B,C are the angles opposing these sides, the Law of Sines states the following

$$\frac{sin\left(\frac{a}{r}\right)}{sin(A)} = \frac{sin\left(\frac{b}{r}\right)}{sin(B)} = \frac{sin\left(\frac{c}{r}\right)}{sin(C)}$$

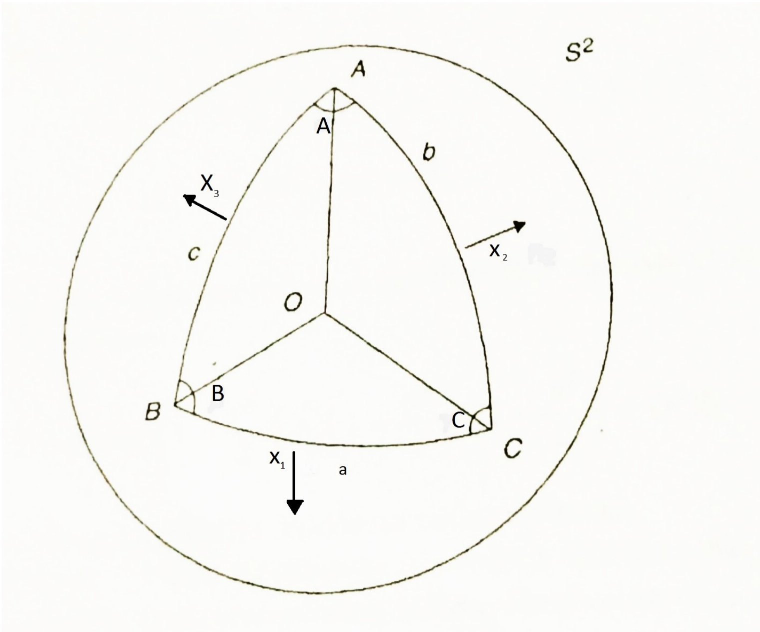
Proof. Let there be a spherical triangle ABC, with radius r. Let $\vec{A}$ be the vector which goes from the origin of the sphere to the vertex A, $\vec{B}$ be the vector which goes from the origin of the sphere to the vertex B, and $\vec{C}$ be the vector which goes from the origin of the sphere to the vertex C Now consider

$$\vec{x}_1 = \frac{\vec{C} \times \vec{B}}{r^2 \sin\left(\frac{a}{r}\right)}$$

$$\vec{x}_2 = \frac{\vec{A} \times \vec{C}}{r^2 \sin\left(\frac{b}{r}\right)}$$

$$\vec{x}_3 = \frac{\vec{B} \times \vec{A}}{r^2 \sin\left(\frac{c}{r}\right)}$$

Each of these are unit vectors and they also must be perpendicular to the respecting great circles, because of the right hand rule. So x_1 is orthogonal to the entire plane of sector COB. Then x_2 is orthogonal to the entire plane corresponding to sector COA. Also x_3 is orthogonal to the plane lying on sector BOA.



Consider the vector identity below.

$$(\vec{A} \times \vec{C}) \times (\vec{C} \times \vec{B}) = (\vec{C} \cdot (\vec{B} \times \vec{A})\vec{C})$$

First note that,

$$(\vec{A} \times \vec{C}) \times (\vec{C} \times \vec{B}) = \left(r^2 \sin\left(\frac{b}{r}\right) \vec{x}_2\right) \times \left(r^2 \sin\left(\frac{a}{r}\right) \vec{x}_1\right)$$

Thus, by we are left with

$$(\vec{A} \times \vec{C}) \times (\vec{C} \times \vec{B}) = -r^4 \sin\left(\frac{a}{r}\right) \sin\left(\frac{b}{r}\right) (\vec{x}_1 \times \vec{x}_2)$$

To determine the value of $(\vec{x}_1 \times \vec{x}_2)$, first consider that $|\vec{x}_1| = |\vec{x}_2| = 1$, since x_1 and x_2 are unit vectors. Also note that $|\vec{C}| = r$. Next we need to consider the angle between $\vec{x}_1$ and $\vec{x}_2$, recall that each of these angles is perpendicular to its respecting great circles.

Suppose we consider these unit vectors when they are each lying exactly at point C. Because these unit vectors are perpendicular to the whole great circle we can arbitrarily place them at a specific point on that great circle without loss of generality.We can arbitrarily place the unit vectors at a point because they do not depend on location, since they are perpendicular at any place on their respecting planes. Since $\vec{x}_1$ is perpendicular to the great circle corresponding to COB, then we know at vertex C, $\vec{x}_1$ will be exactly tangent to the great circle created by vertices C and A. Note also that $\vec{x}_2$ is perpendicular to the great circle corresponding to AOC. Therefore by similar logic $\vec{x}_2$ will be exactly tangent at vertex C to the great circle which goes through vertices B and C. Therefore these two vectors create an angle that must correspond to the angle created at vertex C. Thus

$$(\vec{x}_1 \times \vec{x}_2) = \frac{(1)(1)\sin(C)}{r}\vec{C}$$

Substituting this value in our previous equation we get,

$$(\vec{A} \times \vec{C}) \times (\vec{C} \times \vec{B}) = -r^3 \sin\left(\frac{a}{r}\right) \sin\left(\frac{b}{r}\right) sin(C)\vec{C}$$

So therefore by the original vector identity we have

$$\vec{C} \cdot (\vec{B} \times \vec{A}) = -r^3 \sin\left(\frac{a}{r}\right) \sin\left(\frac{b}{r}\right) sin(C)$$

From this the following is equivalent,

$$\vec{A} \cdot (\vec{C} \times \vec{B}) = -r^3 \sin\left(\frac{b}{r}\right) \sin\left(\frac{c}{r}\right) sin(A)$$

$$\vec{B} \cdot (\vec{A} \times \vec{C}) = -r^3 \sin\left(\frac{c}{r}\right) \sin\left(\frac{a}{r}\right) sin(B)$$

Also note that the triple product of vectors corresponds to the determinant. From properites in linear algebra we know that determinants do not change when two pairs of rows are swapped, which means it is invariant. Therefore, since we are only swapping two pairs of vectors; we know that the equations above all equivalent.

$$\sin\left(\frac{a}{r}\right)\sin\left(\frac{b}{r}\right)sin(C) = \sin\left(\frac{b}{r}\right)\sin\left(\frac{c}{r}\right)sin(A) = \sin\left(\frac{c}{r}\right)\sin\left(\frac{a}{r}\right)sin(B)$$

Then dividing this whole expression by $\sin\left(\frac{a}{r}\right)\sin\left(\frac{b}{r}\right)\sin\left(\frac{c}{r}\right)$, we have

$$\frac{sin(C)}{\sin\left(\frac{c}{r}\right)} = \frac{sin(A)}{\sin\left(\frac{a}{r}\right)} = \frac{sin(B)}{\sin\left(\frac{b}{r}\right)}$$

which equivalently results in

$$\frac{sin\left(\frac{a}{r}\right)}{sin(A)} = \frac{sin\left(\frac{b}{r}\right)}{sin(B)} = \frac{sin\left(\frac{c}{r}\right)}{sin(C)}$$

as desired. □

Now that we have the Pythagorean Theorem, Law of Cosines, and the Law of Sines for a sphere, we can compare these to their planar counterparts. We can consider the following comparison.

Spherical (Let r=1)

Pythagorean Theorem:

$$cos(c) = cos(a)cos(b)$$

Cosines:

$$cos(c) = cos(a)cos(b)+sin(a)sin(b)cos(C)$$

Sines:

$$\frac{sin(a)}{sin(A)} = \frac{sin(b)}{sin(B)} = \frac{sin(c)}{sin(C)}$$

Planar

Pythagorean Theorem:

$$c^2 = a^2 + b^2$$

Cosines:

$$c^2 = a^2 + b^2 - 2abcos(C)$$

Sines:

$$\frac{a}{sin(A)} = \frac{b}{sin(B)} = \frac{c}{sin(C)}$$

Now it is interesting to see that the Law of Sines is quite similar for both planar and spherical. In contrast the Pythagorean Theorem and Law of Cosines are quite different.It is also interesting to note that in both planar and spherical the Pythagorean Theorem is a limit of the Law of Cosines. For the spherical equations this seems natural since we must use the Pythagorean Theorem in order to prove the Law of Cosines; as shown in the proof above. It is important that we recognize the fact that spherical triangles and planar triangles are very different and the comparison of these equations help to show us the differences.

Example 0.0.17. Law of Sines Application

There are many examples we can use for the application for the Law of Sines. To keep variety I will including an example which relates to astronomy. Suppose that we have three lunar rovers, call them A,B, and C respectively. Now suppose we wish to find the distance between two of these such rovers. We can use the following given information and the Law of Sines to discover a particular length.

Radius of moon:1737 km

Distance between rover C and B (length a): 10 km

Angle created at rover B: $\frac{3\pi}{4}$

Angle created at rover A: $\frac{\pi}{6}$

How far are rovers A and C (length of b)?

$$\frac{\sin(\frac{a}{r})}{\sin(A)} = \frac{\sin(\frac{b}{r})}{\sin(B)}$$

Now we can substitute the numerical values and solve for b so we have,

$$b = 1737 \arcsin\left(\frac{\sin(\frac{3\pi}{4})\sin(\frac{10}{1737})}{\sin(\frac{\pi}{6})}\right)$$

$$b = 14.22\ km$$

Thus the distance between rover A and rover C is around 14 km. This value does not take into account any variation, such as dips or holes on the moon's surface so it is best used as a rough estimation. Again it was quite simple to find and gives some idea of how far these lunar rovers are from eachother.

Theses are just a few of the many examples spherical trigonometry has in our world. It is very useful in the study of astronomy and in any navigational problems on our Earth. Having the ability to relate the spherical Phythagoream Theorem, Law of Cosines, and Law of Sines to the radius of a sphere allows us use these equations on spheres of different sizes. This also gives us more freedom in the use of these equations as a whole, over various disciplines such as math, physics, and astromony. Deriving the spherical trigonometric formulas in terms of the radius, was influential in history and its applications and uses continue to the modern day.

Bibliography

[1] Brummelen, Glen Van. *Heavenly Mathematics: The Forgotten Art of Spherical Trigonometry.* Princeton: Princeton UP, 2013. Print.

[2] Clark, C. E. *Proofs of the Fundamental Theorems of Spherical Trigonometry.* Mathematics Magazine 21.3 (1948): 151. Web.

[3] McCleary, John. Geometry from a Differentiable Viewpoint. Cambridge: Cambridge UP, 1994. Print.

[4] Wilson, P. M. H. Curved Spaces: From Classical Geometries to Elementary Differential Geometry. Cambridge: Cambridge UP, 2008. Print.