Lindemann-Weierstrass Theorem With a Bit of π

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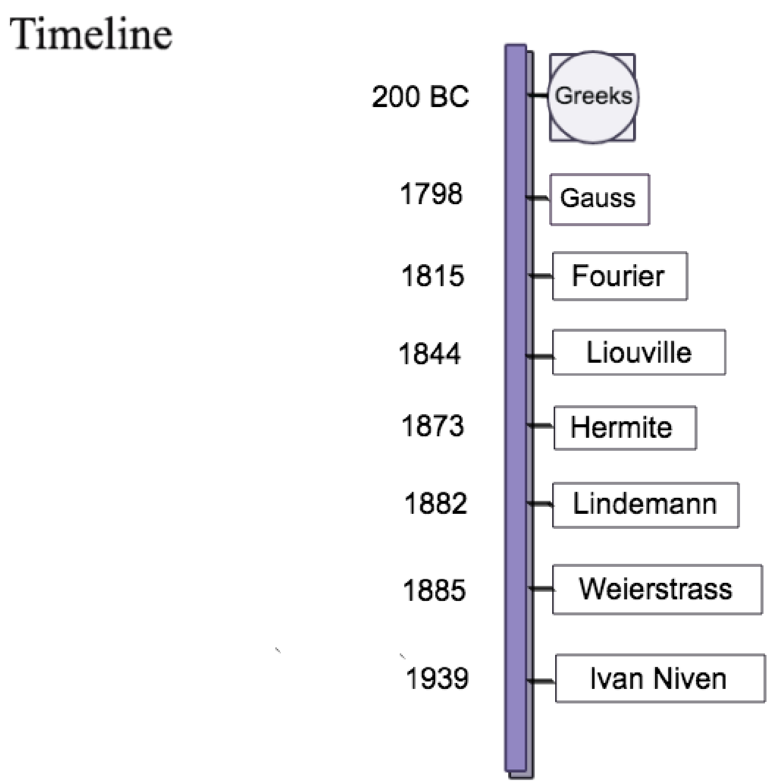
Introduction

Before beginning and diving right into the subject matter, the Lindemann-Weierstrass Theorem, we provide a timeline as well as a table of contents with a series of major events leading up to the theorem:



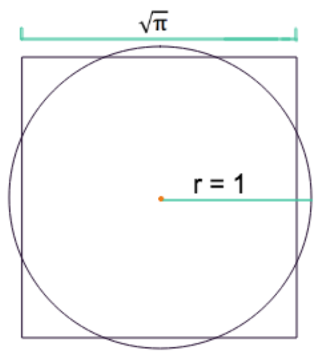
Classical Greek Mathematics

Around 200 B.C. the Greeks had posed the following:

[5] If we are to be given an arbitrary circle, let us define our unit of measure to be the radius of the circle. This means that the circle has area π. With this being said, in order to construct a square of equal area provided we are to only use a straightedge and a compass, we need to construct the side of the square, which must have length $\sqrt{\pi}$.

Which leads us to the following the question that we will answer:

Is it possible for an arbitrary circle to be "squared" provided we are to only use a straightedge and a compass?



The figure above is a representation of an arbitrary circle with an area of π. The square is a representation that has $\sqrt{\pi}$ as its sides to have the same area as the arbitrary circle.

It was not until roughly the middle of the 1800's that Joseph Liouville proved the existence of a large class of numbers that are not "algebraic" (we will define this in the next section). Liouville was able to give insight to a new area in mathematics that the Greeks did not have knowledge of at the time. Following Liouville's discovery in mathematics, Lindemann proved π belongs to this class of numbers which then provided the necessary information that answered the Greek's question.

Johann Carl Friedrich Gauss

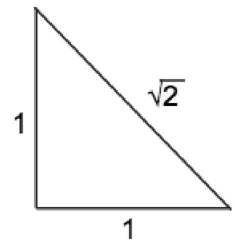
[6] To begin, we will start with Johann Carl Friedrich Gauss (1777-1855). Gauss was a German mathematician who has is referred to as the prince of *all* mathematics. Gauss had a passion for differential geometry. Gauss proved the connection between constructible number (defined below); and algebraic numbers in 1798.

The following definitions needed in order to proceed with the following mathematical findings.

Definition: [1] A *constructible number* is a number which can be represented by a finite number of additions, subtractions, multiplications, divisions, and finite square root extractions of integers.

Such numbers correspond to line segments which can be constructed using only a straightedge and a compass.

Here is an example of constructible number:



Suppose, using a straightedge we construct a right triangle as shown above with the sides being length 1. This means that the hypotenuse of the right triangle will be $\sqrt{2}$. We can confirm this using Pythagorean's theorem as follows

$$a^2 + b^2 = c^2$$

$$1^2 + 1^2 = (\sqrt{2})^2$$

$$1 + 1 = 2$$

$$2 = 2$$

Definition: [1] If r is a root of a nonzero polynomial equation

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 = 0$$

where the a_i's are integers (or rational numbers) and r satisfies no similar equation degree of less than n, then r is said to be an *algebraic number* of degree n. If r is algebraic of degree n, the unique polynomial $f(x) \in \mathbb{Q}[x]$ of degree n with $a_n = 1$ and $f(r) = 0$ is called the minimal polynomial of r.

Note that if r is a rational number, then r is an algebraic number since r is a root of $x - r = 0$. Also note that the product of algebraic numbers is algebraic.

An example of this would be with the polynomial of degree 2 provided below;

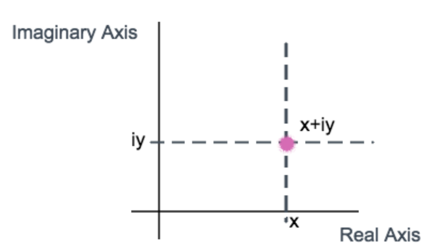
$$5x^2 + \frac{3}{2}x - 4 = 0$$

Since there is not an obvious answer to factoring this equation, we can then use the quadratic formula to find our values of x to be

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-3}{10} \pm \frac{\sqrt{(3/2)^2 - 4(5)(-4)}}{10} \\ &= \frac{-3}{20} \pm \frac{\sqrt{329}}{20} \end{aligned}$$

in this case, the roots would be $\frac{-3+\sqrt{329}}{20}$ and $\frac{-3-\sqrt{329}}{20}$. In this case $\frac{-3+\sqrt{329}}{20}$ and $\frac{-3-\sqrt{329}}{20}$ are both real number and are both algebraic since they are the root of this polynomial equation. Note, many Polynomial equations have solutions that are not real numbers.

Definition: [1] The *complex numbers* are the field $\mathbb{C}$ of numbers of the form $x + iy$ where x and y are real numbers and i is an *imaginary unit* satisfying $i^2 = -1$.



Now, we are able to ask an an immediate observation knowing the definitions before proceeding:

Is i algebraic?

The answer to this, is yes. We can show this in a few easy steps starting with letting i be the root of a polynomial

$$\begin{aligned} x &= i \\ x^2 &= -1 \\ x^2 + 1 &= 0 \end{aligned}$$

Thus showing that i is the root of the polynomial $x^2 + 1 = 0$ of degree 2. Note, both $+i$ and $-i$ are roots of this polynomial. By substituting either of these, we see that both give us the solution we desire.

$$\begin{aligned} (-i)^2 + 1 \\ (-1) + 1 \\ 0 \end{aligned}$$

Given these examples, it is natural to wonder if every complex is algebraic. We will see that the answer is *no*. Leading to the following definition which Jospeh Liouville proved the existence of.

Definition: [1] A *transcendental number* is a (possibly complex) number that is not the root of any integer polynomial, meaning that it is not an algebraic number of any degree. Every real transcendental number must also be irrational, since a rational number is, as noted earlier, an algebraic number of degree one.

Some examples of transcendental number are as listed below.

$$\sum_{n=1}^{\infty} 10^{-n!} = 0.110001000000000000000001 0...$$

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.71828182845904...$$

$$\pi = 3.14159265358979323846264...$$

Now that we know what constructible numbers, algebraic numbers, and transcendental numbers are, we can then refer back to our original questions from the greeks and apply Gauss' logic to bring a closer to what possible answers we can conclude one to be. We can do this by creating the following "if, then" statement:

If $\sqrt{\pi}$ is constructible, *then* we know that $\sqrt{\pi}$ is algebraic. Since the product of two algebraic numbers is an algebraic number then $(\sqrt{\pi})(\sqrt{\pi}) = \pi$ is algebraic. This implies that, by definition, π cannot be transcendental.

Equivalently: *If* π is transcendental, *then* this means that $\sqrt{\pi}$ is not constructible.

To summarize, if $\sqrt{\pi}$ is a constructible number, then this means that a square can in fact have the same area as an arbitrary circle. If this is not the case, and discover π to be transcendental, then it is not possible for a circle to be constructed into a square with the same area provided that one can only use a straightedge and a compass.

Liouville's Theorem

[7] Joseph Liouville (1809-1882) was a French mathematician who established a well-respected journal of mathematics called *Journal de Mathmatiques Pures et Appliques* in which he published some of his own work. Liouville proved the existence of transcendental numbers and demonstrated the first transcendental, *Liouvilles number* in 1844. This is a crucial step forward in the direction needed to solve and answer the Greeks question. Liouville provided a new way to prove a number is not constructible-by proving it is transcendental.

Theorem: (Liouville) [3] Given a real algebraic number α of degree $n > 1$, there is a positive constant $c = c(\alpha)$ such that for all rational numbers $\frac{p}{q}$ with $(p, q) = 1$, $q > 0$, we have

$$|\alpha - \frac{p}{q}| > \frac{c(\alpha)}{q^n}$$

Conceptually, this theorem says that there is a limit to how well an algebraic number can be approximated by a rational number. We can thus prove a number is transcendental by exhibiting a rational approximation that is *too good.*

Below, we will use the theorem to prove Liouville's number is transcendental. First, the proof of Liouville's Theorem.

Proof: [3] Let $P(x)$ be the minimal polynomial of α. By clearing the denominators of the coefficients of $P(x)$, we can get a polynomial of degree n with integer coefficients which is irreducible in $\mathbb{Z}[x]$ and has all positive leading terms. Let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$$

where $f(x) \in \mathbb{Z}[x]$, be this polynomial. We can refer to this as the minimal polynomial of α over $\mathbb{Z}$. Then

$$\left| f(\alpha) - f\left(\frac{p}{q}\right) \right| = \left| \frac{a_n p^n + a_{n-1} p^{n-1} q + ... + a_1 p q^{n-1} + a_0 q^n}{q^n} \right| \geq \frac{1}{q^n}$$

Since $\alpha = \alpha_1, ..., \alpha_n$ are all the roots of $f(x)$,

$$(fx) = a_n (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$$

So

$$\left| f\left(\frac{p}{q}\right) \right| = |a_n| \cdot \left| \frac{p}{q} - \alpha_1 \right| \cdot \left| \frac{p}{q} - \alpha_2 \right| \cdots \left| \frac{p}{q} - \alpha_n \right|$$

by the above $\left| f\left(\frac{p}{q}\right) \right| \geq \frac{1}{q^n}$

let M be the maximum of the values $|a_i|$ is $\left|\frac{p}{q}\right| > 2M$, then

$$\left| \frac{p}{q} - \alpha \right| \geq M \geq \frac{M}{q^n}.$$

If $\left|\frac{p}{q}\right|$ is less than or equal to $2M$, then

$$\left| a_i - \frac{p}{q} \right| \leq 3M$$

that way

$$\left|\alpha - \frac{p}{q}\right| = \frac{\left|f\left(\frac{p}{q}\right)\right|}{|a_n| \prod_{j=2}^{n} \left|\alpha_j - \frac{p}{q}\right|} \geq \frac{1}{|a_n| q^n \prod_{j=2}^{n} \left|\alpha_j - \frac{p}{q}\right|} \geq \frac{1}{|a_n|(3M)^{n-1}}$$

Thus choosing

$$c(\alpha) = \min\left(M, \frac{1}{|a_n|(3M)^{n-1}}\right). \qquad \square$$

Liouville's Number is the first transcendental number to be proven from this theorem.

$$\sum_{n=1}^{\infty} 10^{-n!} = 0.110001000000000000000001 0...$$

Here is an application, let $\alpha = \sum_{n=1}^{\infty} 10^{-n!}$

Proof: [2] Suppose $|\alpha - \frac{p_k}{q_k}| < \frac{c}{10^{(k+1)!}}$ for some constant $c > 0$.

Assuming the claim listed above, and supposing α algebraic of degree j, then by Liouville, there exists a constant γ such that

$$\left|\alpha - \frac{p_k}{q_k}\right| > \frac{\gamma}{10^{k!} j}$$

j is fixed. So by taking k large enough,

$$10^{(k+1)!} > 10^{k!} j$$

Note: $(k+1)! = k!(k+1)$

So for k large enough, (much bigger than j)

$$\frac{\gamma}{10^{k!j}} < \left|\alpha - \frac{p_k}{q_k}\right| < \frac{c}{10^{(k+1)!}}. \square$$

How do we prove a number α is transcendental?

There are many ways that one can go about proving a number is transcendental but, for our purposes, we will use proof by contradiction. The steps to outline how the proofs in my thesis will proceed in the following steps listed in this order. [2]

1. Assume α is algebraic.

By assuming it is algebraic we can then form our number α into a rational number, i.e. $\frac{a}{b}$ where a an b are some positive integers.

2. Build an integer N.

By finding the lowest common denominator, we can then clear the denominators thus giving us N.

3. Show that $N > 0$.

We show that N is greater than 0.

4. Give an upper bound for $N < 1$.

But we can also show its a positive integer less than 1.

5. Apply the Fundamental Principle of Number Theory.

6. Contradiction.

Jean Baptiste Joseph Fourier

[8] Jean Baptiste Joseph Fourier (1768-1830), a French mathematician proved e is irrational in 1815.

Theorem: The number e is irrational.

Although, this is not strictly necessary for proving π to be transcendental, this argument provides a nice illustration of the process just described.

Proof: [2] Suppose $\sum_{n=0}^{\infty} \frac{1}{n!} = e = \frac{a}{b}$ is rational, where a and $b \neq 0$ integers. Stopping the infinite series for e at $n = b$, consider

$$\sum_{n=0}^{b} \frac{1}{n!}$$

Then

$$\frac{a}{b} - \sum_{n=0}^{b} \frac{1}{n!}$$

is positive. Since $b!$ the least common multiple, we can then multiply both sides by $b!$ to obtain a positive integer in which we will call N.

$$\begin{aligned}
N &= b!\left(\frac{a}{b} - \sum_{n=0}^{b} \frac{1}{n!}\right) \\
&= b!\left(\sum_{n=0}^{\infty} \frac{1}{n!} - \sum_{n=0}^{b} \frac{1}{n!}\right) \\
&= b!\left(\frac{1}{(b+1)!} + \frac{1}{(b+2)!} + \frac{1}{(b+3)!} + ...\right) \\
&= \frac{1}{1+b} + \frac{1}{(b+2)(b+1)} + \frac{1}{(b+3)(b+2)(b+1)} + ...
\end{aligned}$$

Using the geometric series, we can then bound the positive integer N

$$\begin{aligned}
N &= \frac{1}{1+b} + \frac{1}{(b+2)(b+1)} + \frac{1}{(b+3)(b+2)(b+1)} + ... \\
&< \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ... = 1
\end{aligned}$$

This is a contradiction since we know that there are no integers between zero and one. Thus, e is irrational. $\square$

Charles Hermite

[9] Charles Hermite (1822-1901) was born in France, and proved e (Euler's number) is a transcendental number in 1873.

Theorem: (Hermite's) The number e (Euler's number) is transcendental.

Lemma: [3] We can notice that for a polynomial f and a complex number t

$$\int_0^t e^{-u} f(u) du = [-e^{-u} f(u)]_0^t + \int_0^t f'(u) du$$

sing integration by parts, and integrating along the line segment joining 0 and t. If we define

$$I(t, f) := \int_0^t e^{t-u} f(u) du,$$

then we see that

$$I(t, f) = e^t f(0) - f(t) + I(t, f').$$

For the sake of notation, we let

$$G(x) = f(x) + f'(x) + f''(x) + ... + f^{(m)}(x)$$

where $f(x) \in Q[x]$ and the degree of f.

Iterating the relation $(*)$ we have

$$I(t,f) = e^t \sum_{j=0}^{m} f^{(j)}(0) - \sum_{j=0}^{m} f^{(j)}(t) = e^t G(0) - G(t)$$

By definition of the contour integral, parametrizing the segment from 0 to t by $\gamma(s) = st$, we have

$$\begin{aligned} I(t,f) &= \int_0^t e^{t-u} f(u) du \\ &= \int_0^1 e^{t-st} f(st) t ds \\ &= \int_0^1 t e^{(1-s)t} f(st) ds \end{aligned}$$

Then, by the Triangle Inequality Theorem

$$\begin{aligned} |I(t,f)| &= \left| \int_0^1 t e^{t-st} f(st) ds \right| \\ &\leq \int_0^1 |t e^{(1-s)t} f(st)| ds \\ &\leq |t| \int_0^1 |e^{(1-s)t} f(st)| ds \end{aligned}$$

If we let $M = \max_{0 \leq s \leq 1} |e^{(1-s)t} f(st)|$ then

$$
\begin{aligned}
|I(f,t)| &\leq |t| \int_0^1 M ds \\
&\leq |t| M \cdot \text{Length}[0,1] \\
&\leq |e^{(1-s)t}| (|a_m||st|^m + |a_{m-1}||st|^{m-1} + \cdots + |a_1||st| + |a_0|)
\end{aligned}
$$

Let $F(x) = |a_m|x^m + |a_{m-1}| + \cdots + |a_1|x + |a_0|$. Looking at the maximum, we apply the triangle inequality again and can rewrite

$$
\begin{aligned}
&= |e^{1-s)t}| \\
&= |e^{(1-s)t}| \cdot F(|st|) \text{ where } 0 \leq s \leq 1
\end{aligned}
$$

We then take the minimum of s for $|e^{(1-s)t}|$ and the maximum of $F(|st|)$

$$\leq e^{|t|} F(|t|)$$

then it is easy to see from the definition of $I(t,f)$ that

$$|I(t,f)| \leq |t| e^{|t|} F(|t|).$$

□

Proof: (Hermite) [3] Suppose e is algebraic of degree n. Then there exists a polynomial

$P(x) \in \mathbb{Z}[x]$ such that

$$P(e) = a_n e^n + a_{n-1} e^{n-1} + ... + a_1 e + a_0 = 0$$

for some integers a_i and a_0 , $a_n \neq 0$. Let

$$J := \sum_{k=0}^{n} a_k I(k, f)$$

or

$$f(x) = x^{p-1}(x-1)^p ... (x-p)^p$$

where $p > |a_0|$ is a large prime. By $(**)$ in mind, we see that

$$\begin{aligned}
J &= \sum_{k=0}^{n} a_k I(k, f) \\
&= \sum_{k=0}^{n} a_k (e^k G(0) - G(k)) \\
&= \sum_{k=0}^{n} a_k e^k (G(0)) - \sum_{k=0}^{n} a_k (G(k)) \\
&= G(0) \sum_{k=0}^{n} a_k e^k - \sum_{k=0}^{n} a_k (G(k)) \\
&= G(0)(P(e)) - \sum_{k=0}^{n} a_k (G(k)) \\
&= (G(0))(0) - \sum_{k=0}^{n} a_k (G(k)) \\
&= -\sum_{k=0}^{n} a_k (G(0))
\end{aligned}$$

So

$$J = -\sum_{k=0}^{n} a_k(G(0)$$

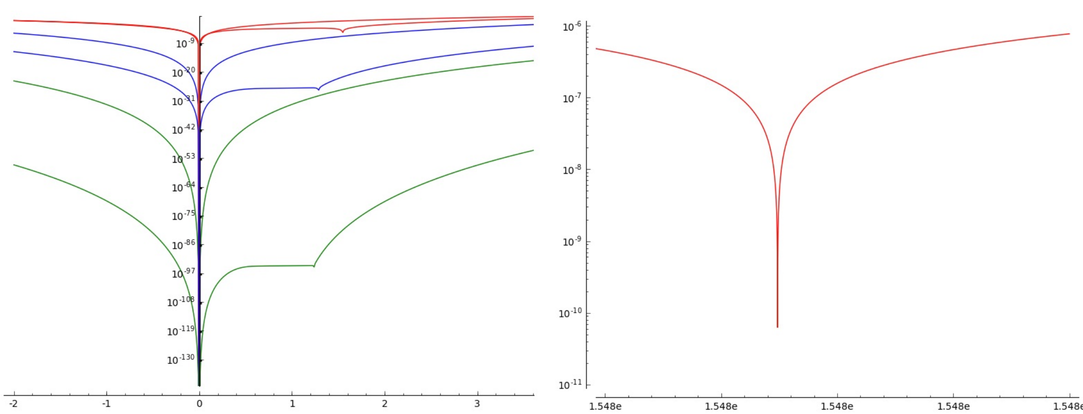
Since f has zero of order p at $1, 2, ..., n$ and a zero of order $p-1$ at 0, we have that the summation for $G(k)$ actually starts from $f^{(p-1)}(k)$. For $j = p-1$, the contribution to $G(k)$ from $f^{(j)}$ is

$$f^{(p-1)}(0) = (p-1)!(-1)^{np} n!^p \text{ for } G(0)$$

for $G(0)$ and 0 for $G(k)$, $k \geq 1$.Thus if $n < p$, then $f^{(p-1)}(0)$ is divisible by $(p-1)!$ but not by p. If $j \geq p$, we see that $f^{(j)}(0)$ and $f^{(j)}(k)$ are divisible by $p!$ for $1 \leq k \leq n$. Hence J is a non-zero integer divisible by $(p-1)!$ and consequently

$$(p-1)! \leq |J|.$$

Notice, the polynomial $P(x)$ provides an approximation to e^x that is not necessarily much better than the Taylor polynomial for most values of x, but which for large primes p is incredibly good at $x = 0, 1, 2, 3, ..., d$.



error in the approximation for case $d = 1$.

In the figure on the left, there are different values of p (3,11, 17) being graphed in comparison to the Taylor polynomial. It is easy to see that for a larger value of p the smaller the error in approximation becomes in comparison to to the Taylor polynomial. However, as we move away from the origin, it is no better than the Taylor polynomial. In the figure on the left, is a zoomed in image of the small dip when $p = 3$, where this dip happens, is when the error becomes extremely small for that value.

One the other hand, our estimate for $|I(t, f)| \leq |t|e^{|t|}F(|t|)$ shows that

$$|J| \leq \sum_{k=0}^{n} |a_k|e^k F(k)k \geq Ane^n(2n)!^p$$

where A is the maximum of the absolute values of the $|a_k|$'s. The elementary observation

$$e^p \leq \frac{p^{p-1}}{(p-1)!}$$

gives

$$p^{p-1}e^{-p} \leq (p-1)! \leq |J| \leq Ane^n(2n)!^p$$

For p sufficiently large, this is a contradiction. $\square$

Carl Louis Ferdinand von Lindemann

[10] Ferdinand Von Lindemann (1852-1939) was born and raised in Germany. He spent most of his area of studies in geometry and Analysis. Ferdinand was heavily influenced by Hermite and proof of e being transcendental and was able to base his proof of π being a transcendental number on Hermite's proof of e being a transcendental number.

Lindemann's Theorem: The number π is transcendental.

Proof: [4] Suppose π is not transcendental. We also know that i is algebraic. Then we can say $\alpha = i\pi$ is also algebraic since we know that products of algebraic numbers are algebraic as well. Let $P(x)$ be the minimal polynomial of α. And have degree $d = \deg P(x)$ and let $\alpha = \alpha_1, ..., \alpha_d$ be all the roots of $P(x)$. Let N be the leading coefficient of the minimal polynomial where N is an algebraic number of x^d in $P(x)$. Just like before, the product $N\alpha$ is an algebraic integer.

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

and the set can be rewritten as polar coordinates as follows

$$\mathbb{C} = \{re^{i\theta} | r \geq 0, -\pi \leq \theta \leq \pi\}$$

Recall that the polar form of a complex number is $z = e^{i\theta}$ where $r > 0$ and $-\pi \leq \theta \leq \pi$.

Recall also that In our case, $\theta = \pi$ so then with substitution we can see that

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$e^{i\pi} = \cos(\pi) + i\sin(\pi)$$

so

$$e^{i\pi} = (-1) + i(0)$$

and thus, $e^{i\pi} = -1$.

Since

$$e^{i\pi} = -1,$$

we know that one of the e^{α_i}'s is equal to $e^{i\pi} = -1$. So the expression

$$(1 + e^{\alpha_1})(1 + e^{\alpha_2}) \cdots (1 + e^{\alpha_d})$$

will be zero because we know that there will be a term in the product sum where

$$(1 + e^{\alpha i}) = (1 - 1) = 0$$

The product $(1+e^{\alpha_1})(1+e^{\alpha_2}) \cdots (1+e^{\alpha_d})$ can be expanded as a sum of 2^d terms of the form e^{θ} where

$$\theta = \epsilon_1\alpha_1 + ... + \epsilon_d\alpha_d$$

where each $\epsilon_i \in \{0, 1\}$.

Suppose that exactly n of these numbers are non-zero and denote them by $\beta_1, ...\beta_n$. Note that these numbers constitute all the roots of a polynomial with integer coefficients. To see this, it suffices to observe that the polynomial

$$\prod_{\epsilon_1=0}^{1} ... \prod_{\epsilon_d=0}^{1} (x - (\epsilon_1\alpha_1 + ... + \epsilon_d\alpha_d))$$

is symmetric in $\alpha_1, ..., \alpha_d$ and hence lies in $\mathbb{Q}[x]$. The roots of this polynomial are $\beta_1, ..., \beta_n$ and 0 which has multiplicity $a - 2^d - n$. Dividing by x^a and clearing the denominator, we get a polynomial in $\mathbb{Z}[x]$ with roots $\beta_1, ..., \beta_n$. Now

$$(1 + e^{\alpha_1})(1 + e^{\alpha_2})...(1 + e^{\alpha_d}) = 0$$

which implies

$$(2^d - n) + e^{\beta_1} + ... + e^{\beta_n} = 0$$

with $I(t, f)$ as in the previous chapter, we consider the combination

$$K := I(\beta_1, f) + .. + I(\beta_n, f)$$

where

$$f(x) = N^{np}x^{p-1}(x - \beta_1)^p...(x - \beta_n)^p$$

we have $m = (n+1)p-1$ and p denotes again a large prime. Again denoting $G(x) = \sum_{j=0}^{m} f^{(j)}(x)$ thus,

$$K = -(2^d - n)\sum_{j=0}^{m} f^{(j)}(0) - \sum_{j=0}^{m}\sum_{k=1}^{n} f^{(j)}(\beta_k)$$

.

$$K = -(2^d - n)G(0) - \sum_{k=1}^{n} G(\beta_k)$$

For each $j \geq 0$,The sum over k is a symmetric function in $N\beta_1, ..., N\beta_n$ since these are all the roots of a monic polynomial over the integers, we conclude that $\sum_{k=1}^{n} f^{(j)}(\beta_k)$ is an integer.

Before proceeding, we will introduce a lemma before proceeding.

Lemma 2 : Let $z_1, ..., z_n \in \mathbb{C}$ such that $z_1, ..., z_n$ are all the roots of a monic polynomial $P(x)$ with integer coefficients. Then every symmetric polynomial with integer coefficients in $z_1, ..., z_n$ is an integer.

Proof: First, recall that

$$P(x) = \prod_{k=1}^{n}(x - z_k) = x^n - e_1(z_1, ..., z_n)x^{n-1} + e_2(z_1, ..., z_n)x^{n-2} + \cdots + (-1)^n e_n(z_1, ..., z_n)$$

where $e_1, ..., e_n$ are the elementary symmetric functions. Since $P(x) \in \mathbb{Z}[x]$, $e_1, ..., e_n$ are integers. Then by the Fundamental Theorem of Symmetric Functions, ever symmetric polynomial with integer coefficients in the $z_1, ..., z_n$, is a polynomial in the $e_1, ..., e_n$ (with integer coefficients), hence the symmetric polynomial is an integer. $\square$

Continuing with the proof of the number π is transcendental;

For ear fixed j, $0 \leq j \leq m$,

$$f^{(j)}(\beta_1) + f^{(j)}(\beta_2) + \cdots + f^{(j)}(\beta_n)$$

is a symmetric polynomial in $z_1 = N\beta_1$, $z_2 = N\beta_2$,...,$z_n = N\beta_n$. It follows from what we proved above in Lindemann's proof that $z_1, ...z_n$ are all root of an integer polynomial. Thus, by lemma, the sum $(*)$ is an integer. Hence $\sum_{k=1}^{n} G(\beta_k)$ is an integer. Similarly, each $f^{(j)}(0)$ is symmetric in $z_1, ..., z_n$ hence each $f^{(j)}(0)$ is an integer. It follows that

$$|K| = \left| -(2^d - n)G(0) - \sum_{k=1}^{n} G(k) \right|$$

is a non-negative integer.

$f^{(j)} = 0$ for $j < p-1$ and $f^{(p-1)}(0) = (p-1)!(-N)^{np}(\beta_1 \cdots \beta_n)^p$.

For $j \geq p$, $f^{(p)}(0)$ is divisible by $p!$.

$f^{(j)}(\beta_k) = 0$ for $j < p$,

$$\sum_{k=1}^{n} f^{(j)}(\beta_k)$$

(which you will recall from above is an integer) is divisible by $p!$ for $j \geq p$.

The facts listed above imply that $|K|$ is divisible by $(p-1)!$, hence $0 < |K|$.

We can then say

$$N = \frac{|K|}{(p-1)!}$$

where N is a positive integer.

By the $|I(f,t)| \leq |t|e^{|t|}F(|t|)$ lemma,

$$|K| \leq \sum_{k=1}^{n} |\beta_k| e^{|\beta_k|} F(|\beta_k|).$$

Let $B = \max\limits_{1 \leq k \leq n} |\beta_k|$. Then

$$|K| \leq \sum_{k=1}^{n} Be^B F(B) = nBe^B F(B).$$

Recall that, $f(x) = N^{np}x^{p-1}(x+\beta_1)^p \cdots (x+\beta_n)^p$ where $f(x) \in \mathbb{Z}[x]$ and as in the proof in lemma 2, $f(x) = N^{np}x^{p-1}(x^n + e_1x^{n-1} + e_2x^{n-2} + \cdots + (-1)^n e_n)^p$. So $F(x) = |N|^{np}x^{p-1}(x^n + |e_1|x^{n-1} + |e_2|x^{n-2} + \cdots + |e_n|)^p$ and hence

$$\begin{aligned} F(B) &= |N|^{np}B^{p-1}(B^n + |e_1|B^{n-1} + \cdots + |e_n|)^p \\ &\leq [|N|^n B(B^n + |e_1|B^{n-1} + \cdots + |e_n|)]^p \end{aligned}$$

In conclusion, with Liouville's discovery in mathematics, Lindemann was able to prove π to be transcendental which then answers the Greek's questions. Since π is transcendental, $\sqrt{\pi}$ is not constructible. In other words, it is not possible to construct *pi* using only a straightedge and a compass. After Lindemann's proof was published, other later on, like Karl Weierstrass, further extended his theorem as well as went on to simplify his proof.

Karl Weierstrass

[11] Karl Weierstrass (1815-1897) is known and referred to as "the father of modern analysis." Weierstrass derived an extension to Lindemann's proof of π being transcendental to help support Lindemann's findings. The theorem is referred to as the *Lindemann-Weierstrass theorem.*

Theorem: [3] **(Lindemann-Weierstrass Theorem**

If $\alpha_1, ..., \alpha_s$ are distinct algebraic numbers, then $e^{\alpha_1}, ..., e^{\alpha_s}$ are linearly independent over $\overline{\mathbb{Q}}$, where $\overline{\mathbb{Q}}$ is the field of algebraic numbers.

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